Polarization in Multi-level Elections

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Abstract

We develop a spatial model which embeds regional- and national-level elections in a single theoretical framework, and links the two via the assumptions that: a.) voters punish parties for intraparty platform disunity, and b.) regional candidates value having their party in national office. Although candidates are purely office-seeking, polarization occurs when platforms are captured by regional extremists. Interestingly, it is in 'decentralized' systems that national politicians can escape this centrifugal force, and adopt the national median as their platform. Conversely, party capture only occurs in sufficiently 'centralized' systems, where pressures for party unity generate incentives for regional pandering. In centralized systems, platform polarization depends in surprising and non-monotonic ways on geographic polarization in the electorate: decreasing geographic polarization at times leads to increased polarization of political parties. In sufficiently centralized systems, a form of asymmetric party system polarization may emerge *despite* the presence of a moderate electorate.

1 Introduction

A cumulative body of empirical work on economic and political geography shows that ideological polarization has a geographic gradient. The concentration of radical right vote in the so-called 'left-behind' places is but one example of a well-understood phenomenon: ideologies tend to cluster in space, within but also between regions (Rodden, 2010; Rodríguez-Pose, 2018; Cremaschi et al., 2024). One need look no further than the United States, United Kingdom, and Spain to find political systems in which voters' ideological preferences cluster along regional lines; and in which political conflict acquires a strongly geographic and sometimes secessionist dimension. The issue is not new, but as economic prosperity becomes more and more concentrated in clusters of global cities, spatial tensions and their implications for democracy are re-taking center stage (Rodden, 2019; Beramendi and Rogers, 2022; Iversen and Soskice, 2019). What is relatively new today is the intensity of polarization in space and its political implications.

This paper's formal model provides a nuanced understanding of the multi-dimensional nature of contemporary political polarization, and the types of political institutions which might channel this polarization most effectively. It contains two basic design characteristics, presented in Section 2. Firstly, it embeds regional- and national-level elections in a single model. Secondly, it links the outcomes in these elections via two assumptions: a) that regional candidates receive a utility bonus if their party wins the national contest, and b) that voters punish parties for internal platform disunity. With this set-up, we identify the conditions under which regional extremists are able exert centrifugal pressure on party systems; and expose the oftentimes surprising relationship between baseline geographic polarization in the electorate and the equilibrium polarization of party platforms.

Two key political geography parameters determine the game's outcomes. The first is the size of the bonus that regional candidates derive from winning the national contest. This parameter should reflect the relative fiscal, regulatory, and judicial power of national as compared to regional governments. The higher the relative power of the center across these policy realms, the larger should be the bonus obtained by regional candidates when their party wins the national election. With a certain degree of conceptual liberty (see below), we thus label this parameter a political system's *centralization*. The second parameter of interest is the distance between the median voters from different regions, defined as the electorate's baseline level of *geographic polarization*.

The model yields three central insights. First, and somewhat paradoxically, it is in decentralized systems that national politicians can escape the centrifugal forces of regional extremism. The mechanism is as follows: in decentralized systems, regional candidates are primarily concerned with winning locally, and thus choose their regional median voter's ideal point as a platform. This in turn stabilizes national election platforms at the national median, as any attempt to create greater party unity in one region would increase disunity in the other. The resulting equilibrium is labeled *Geographic Balancing*, and is characterized by two properties: a) heterogeneous parties, in which candidates represent the interests of distinct regional constituencies, and b) national-level moderation, insofar as national candidates become anchored the national median.

Second, and once again paradoxically, it is only in sufficiently centralized systems that regional extremists may capture political parties. The key strategic mechanism is the premium that centralized systems place on maintaining party unity. When regional candidates place high value on control over the national government, they may forgo incumbency in their own region and pander to another region's more extreme voters. In turn, to maintain party unity national candidates may be forced to follow suit, leading to polarized equilibria in which parties are highly unified but ideologically disparate. Note that candidates are purely office-seeking: polarization results *not* from the fact that national candidates have non-centrist policy preferences, but rather from party capture by regional extremists.

Our third core insight is that the relationship between baseline geographic polarization and equilibrium party system polarization depends on the extant level of centralization, and is often non-monotonic. In decentralized systems national parties remain anchored at the national median regardless of the level of geographic polarization in the electorate. When centralization is intermediate, party system polarization bears a 'hump-shaped' relationship with baseline geographic polarization, reaching its highest levels when geographic polarization is intermediate. When centralization is high, party system polarization bears a 'U-shaped' relationship with baseline geographic polarization, reaching its highest levels when geographic polarization is either low or high. Indeed, in sufficiently centralized systems, low levels of polarization in the electorate may nonetheless co-exist with high levels of *Asymmetric Polarization* in the party system, with one party unified at the center and the other at a regional extreme.

Beyond these substantive findings, the paper provides a basic theoretical result on party competition in multi-level elections. This result, stated in Theorem 1, tells us that despite the game's complexity one of two basic convergence patterns must emerge in equilibrium. The first is Geographic Balancing, in which all candidates converge to the median in their particular electoral contest. This equilibrium format exhibits full convergence *between* parties, but high levels of divergence and heterogeneity *within* parties. The second convergence pattern is Party Unity, in which all candidates (regional and national) from the same party announce the same platform, and in which the two parties announce distinct platforms. This equilibrium format exhibits full convergence *within* parties, but often high levels of divergence *between* parties in the form of political polarization. The Conclusion discusses further applications of this basic result.

1.1 Related Literature

The paper is connected to three distinct strands of literature in formal political economy: a.) that on multi-district or simultaneous elections; b.) that on endogenous valence, and c.) that on federalism and decentralization. Regarding (a), Austen-Smith (1981), Callander (2005), and Bernhardt et al. (2020) analyze models in which parties compete in n districts, and a national leader must choose a single spatial platform to run in all districts. In these papers, there are multiple local contests but no strategic local candidates. In contrast, Austen-Smith (1984) analyzes a model in which local candidates announce district-level platforms; and parties' national platforms are an *aggregate* of these individual candidate decisions. Like Austen-Smith (1984), Krasa and Polborne (2018) assume that party platforms are an aggregate of local candidate platforms, but add the assumption that voters are 'strategic': they only vote based on national-level considerations if their district has some chance of electing the 'pivotal' (majority-determining) legislator.[1] Eyster and Kittsteiner (2007) and Zhou (2025) develop multi-district models in which *both* national party leaders *and* local candidates must simultaneously announce spatial platforms. In the former local candidates pay a linear campaign cost when deviating from their national party's platform; and in the latter voters partly infer local candidate platforms from national party platforms (and vice versa).

While the above papers model multi-district elections, Alesina and Rosenthal (1996) and Persson et al. (1997) study multi-branch elections, in which voters simultaneously cast votes for two distinct political offices – the Legislature and the Presidency. Alesina and Rosenthal (1996) show that centrist voters often have the incentive to split their ticket and create divided government, as legislative bargaining then leads to centrist policies. Persson et al. (1997) show that a separation of powers system may generate high levels of political accountability, but only if the right post-election bargaining rules (i.e. checks-and-balances) are in place.

A second line of research connected to the current paper develops formal models of endogenous valence, where valence captures a 'quality' dimension (e.g. competence or charisma) that is orthogonal to spatial platforms (Groseclose, 2001; Aragones and Palfrey, 2002). Carrillo and Castanheira (2008), Ashworth and De Mesquita (2009), Zakharov (2009), and Serra (2010) all develop models in which valence is determined endogenously by a candidate's investment of costly effort in reputation-building. A second set of papers studies how campaign spending tech-

[1] Relatedly, Polborne and Snyder (2017) assume that voters in all districts infer a party's national policy from the platform of its *median elected legislator*.

nologies (Iaryczower and Mattozzi, 2012, 2013; Balart et al., 2022) and party primaries (Crutzen et al., 2010) can serve as direct sources of endogenous valence. While the specific mechanism and general model are very different, our approach is closest to that in Izzo (2023), where criticism of party leaders by disaffected factions reduces a party's perceived quality.

A third line research connected to the current paper develops formal models of federalism and decentralization. Early welfare economics approaches to federalism modeled elections as a tool that provided information about voters' true preferences to rulers, both regional and central, intent on maximizing efficiency in the provision of aggregate welfare (Tiebout, 1956; Oates, 1972; Weingast et al., 2005). Other papers turn their attention to institutional design. How can we design federations so as to maximize complementarities and limit free-riding, opportunism, or races to the bottom (Bednar et al., 2009; Weingast, 2014)? Under what conditions do voters prefer higher levels of centralization or prioritize the representation of voters versus territories (Crémer and Palfrey, 1999)? Under what conditions does centralization of economic policy-making promote rent-seeking (Persson and Tabellini, 1994) as opposed to effective risk-sharing (Persson and Tabellini, 1996)?

From this literature, the current paper has the most in common with Dixit and Londregan (1998), which includes both regional and national elections, and examines how the two levels interact in a model of distributive politics. Our paper differs from Dixit and Londregan (1998) in a number of important ways. First, in the latter candidate platforms are fixed exogenously, with the focus being on the endogenous choice of regional transfer schemes. Second, and related, the externalities in Dixit and Londregan (1998) come from the connection between national and local fiscal policy choices; whereas in our paper they come from the costs of intra-party platform disunity, and the bonus accrued by regional candidates for national incumbency. As a result, the two papers ask different questions, and provide different theoretical results and substantive conclusions.

2 The Model

A polity is divided into two regions, denoted $R \in \{1, 2\}$, each with population normalized to 1. Political competition takes place in a continuous spatial dimension x, whose range we restrict for convenience to $x = [0, 2]$. A subnational election is held in each region (e.g. gubernatorial, regional parliaments, etc.), and a national election N is held which involves voters from both regions. All contests are held simultaneously and under plurality rule. There are two political parties $P \in \{A, B\}$ each comprised of three strategic actors: a candidate in region 1, a candidate in region 2, and a candidate in the national election N. Label A's (B's) candidate in region R as A^R (B^R), and A's (B's) national candidate as A^N (B^N).

In the game's first stage all six candidates simultaneously announce a platform in their respective contests. Let x_P^R (x_P^N) be the platform announced by P's candidate in region R (the national election), and define $\mathbf{x} = \left\{x_A^1, x_A^2, x_A^N, x_B^1, x_B^2, x_B^N\right\}$ as a vector of pure strategies. Candidates are office-seeking, and all things equal seek to maximize the probability of winning their own contest. In addition, regional candidates gain a bonus $\beta \geq 0$ in the event that their party wins the national election. This bonus will be higher when control over fiscal policy, regulation, and the judiciary are concentrated at the national level.

Define $\pi_P^R(\mathbf{x})$ as P's probability of winning the contest in region R and $\pi_P^N(\mathbf{x})$ as P's probability of winning the national contest. In turn, utility functions for national candidates (A^N and B^N) regional candidates (A^1, A^2, B^1, B^2) can be written, respectively, as:

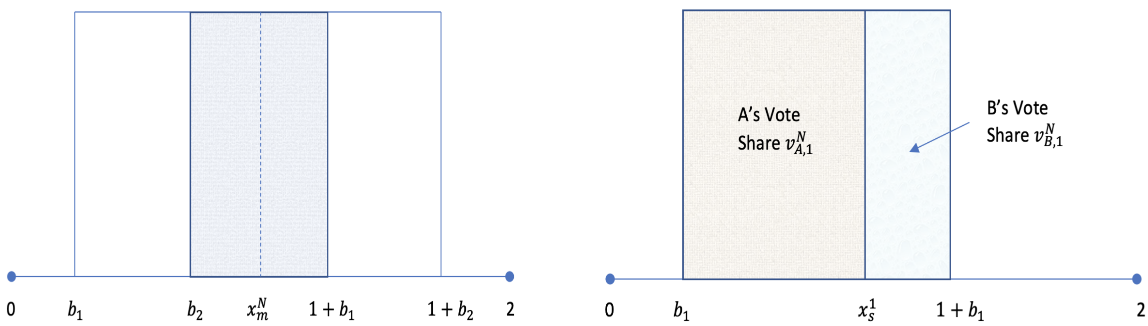
$$U_P^N(\mathbf{x}) = \pi_P^N(\mathbf{x}), \tag{1}$$

$$U_P^R(\mathbf{x}) = \pi_P^R(\mathbf{x}) + \pi_P^N(\mathbf{x}) \cdot \beta. \tag{2}$$

National candidates maximize the likelihood of winning the national contest. As $\beta \to 0$ regional candidates maximize the probability of winning their local race, disregarding the national outcome. On the other hand, as β becomes arbitrarily large, they maximize their party's probability of winning the national contest, disregarding regional outcomes. At intermediate β regional candidates face trade-offs in balancing their desire for both local and national office.

Voter i from region R is characterized by her ideal point in x_i^R, and voter ideal points in R are distributed uniformly over the support set $[b_R, 1 + b_R]$ where $0 \leq b_R \leq 1$. This implies that R's *regional median voter* is located at $x_m^R = b_R + \frac{1}{2}$. Without loss of generality assume $x_m^1 < x_m^2$, i.e. that region 1's median is more 'left-leaning' than region 2's median. Given our distributional assumptions, the two regional medians are positioned symmetrically around the national median, $x_m^N = \frac{x_m^1 + x_m^2}{2}$. Once again without loss of generality, assume that the *national median voter* is located at $x_m^N = 1$, which implies that $x_m^1 = 2 - x_m^2$. Define *baseline polarization* in the electorate as $d = (x_m^2 - x_m^1)$, the ideological distance separating the two regional medians.

Panel (a) in Figure 1 provides a visualization of the model's electoral topology. Note that except for the parametric case in which $x_m^1 = \frac{1}{2}$ and $x_m^2 = \frac{3}{2}$, some portion of the spatial dimension $x \in [0, 2]$ will be uncovered by the electorate. For example, if $x_m^1 = \frac{3}{5}$ and $x_m^2 = \frac{7}{5}$ then there will no voter ideal points in the spaces $x \in [0, \frac{1}{10})$ and $x \in (\frac{19}{10}, 2]$. While we do not restrict candidates *a priori* from choosing platforms in these uncovered spaces, we rule this behavior out in equilibrium below.



(a) A Polity (b) Vote Shares in Region 1

Figure 1: The shading in (a) represents the ideological overlap between regions 1 and 2.

After observing a set of platforms $\mathbf{x}$, all citizens simultaneously cast two votes, one in their regional election and one in the national election. They can split their vote, for example choosing party A's candidate in the regional election but B's in the national election. In regional elections we assume that voters deterministically choose the candidate whose regional platform is closest to their ideal point (and randomize if the two regional platforms are equidistant).

In contrast, voter choice in the national contest is impacted by a *valence shock* and *party disunity penalties.* We capture the former with an exogenous disturbance σ, distributed uniformly over the support set $[-1, 1]$, that affects voter preferences for party A. The value of σ is realized *after* candidates make their platform choices but *before* voters cast their ballots. We capture the latter by assuming that, in the national contest, voters *penalize parties that exhibit internal platform disunity.* In particular, assume that voters in region R penalize national parties by a factor of $-k \cdot \left|x_P^R - x_P^N\right|$: the more distant is the position x_P^R from the position x_P^N, the more voters in region R will penalize P in the national election. The constant $k \in [0, 1]$ represents the weight that voters attach to concerns of party unity.

We choose a linear (absolute value) disunity cost due in part to its numerical simplicity, but also because we have no specific behavioral priors about the shape of this function. Should it be quadratic, like the usual spatial loss function, such that voters are particularly responsive to 'extreme' disunity but less so to moderate disunity? Or should voters be especially responsive to changes from very unified parties to moderately disunified parties? As discussed below, we look forward to future empirical and theoretical research on the sources and shape of voter responsiveness to organizational disunity.

By combining the probabilistic shock σ and the disunity penalty $-k \cdot \left|x_P^R - x_P^N\right|$ with a traditional quadratic loss function, voter i's utility for national candidates A^N and B^N can be written, respectively, as:

$$u_i^A(\mathbf{x}) = -(x_i^R - x_A^N)^2 - k\left|x_A^N - x_A^R\right| + \sigma, \tag{3}$$

$$u_i^B(\mathbf{x}) = -(x_i^R - x_B^N)^2 - k\left|x_B^N - x_B^R\right|, \tag{4}$$

Voters choose A in the national contest if $u_i^A(\mathbf{x}) > u_i^B(\mathbf{x})$, choose B if $u_i^B(\mathbf{x}) >$

$u_i^A(\mathbf{x})$, and randomize if $u_i^A(\mathbf{x}) = u_i^B(\mathbf{x})$. To summarize, the game sequences is:

1. all six candidates adopt platforms in their respective contests;
2. σ is realized, and voters observe both $\mathbf{x}$ and σ;
3. voters choose based on spatial proximity in the regional contest; and choose the candidate that yields the highest utility $u_i^P(\mathbf{x})$ in the national contest;
4. winners in all contests are assigned via plurality rule.

2.1 National Bonuses and Disunity Penalties

Before proceeding to the baseline results, allow us to briefly substantiate the distinction between the bonus regional candidates obtain from their party winning the national election (β), and the size of the penalty incurred as a result of internal party disunity (k). Note first that, from a technical point of view, our model is not incompatible with a certain level of co-determination between the two parameters. We present results below for situations in which both β and k are high, and those in which both β and k are intermediate or low.

More importantly from a substantive point of view, we do not expect the two to be highly co-determined. As elaborated above, β will be primarily a function of the policy influence exercised by the central government on core state functions. These include the regulation of private and public activities, the provision of public goods, fiscal redistribution, and judicial review. Though β does not capture institutions directly, it links the premium regional leaders gain from winning the national election back to the actual distribution of powers within the polity. In the Conclusion, we discuss extensions which build on Cremer an Palfrey (1999) and Dixit and Londregan (1998) in modeling centralization as a weighted function of distinct regions and/or levels of government in the policy-making process.

Voters' penalty for party disunity has different roots. In a growing body of observational and experimental research, internal party conflict has been shown to negatively impact an organization's overall 'valence' assessment (Clark, 2009, 2014; Ceron and Volpi, 2022). Relatedly, in national legislatures, disunity and indiscipline may be interpreted as a signal that national parties will be unable to implement their agendas, and that national party leaders are ineffective. Indeed, internal conflict undermines the perception of policy competence, even on issues where the party had a credible claim to ownership (Greene and Haber, 2015). Though past evidence was mixed, more recent research using a regression discontinuity design suggests that divisive primaries do indeed undermine party performance in US congressional elections (Fouirnaies and Hall, 2019). Lehrer et al. (2022) provide the first experimental evidence that party disunity dampens voter support.[2]

Arguably, concerns about valence and the ability to implement one's policy agenda may be more pressing in some political systems than others. Presidential systems, for example, are particularly prone to ineffectiveness if competing internal factions weaponize opposing branches of power. Similarly, intra-party dissension may lead to concerns of policy ineffectiveness in majoritarian systems (e.g. the UK) or proportional systems with strong majoritarian biases (e.g. Spain), where single-party or dominant-party government is the norm. Voters may be less attuned to factional conflict in pure PR systems, where multi-party governments are the norm, and since extreme factionalism can be resolved by party entry.

As a result, while an exhaustive analysis of the determinants of k is beyond our current scope, we do not believe that β and k are highly co-determined. The empirical evidence reviewed above suggests that voters penalize party disunity in a wide variety of institutional contexts, including in more decentralized federal states such as the United States and Germany. Conversely, one can imagine systems with higher levels of β in which voters discount intra-party divisions for one or another institutional or contextual reason. The quest for k's determinants, and more generally for the 'shape' of voter responses to organizational disunity, represents a promising area for future research.

[2] They decompose intra-party conflict in three elements, namely public critiques, (mis)behavior at party conferences, and breaches of party discipline in the legislature. All three emerge as strong predictors of reductions in electoral support in a conjoint experiment deployed in Germany.

3 Baseline Results

Recall from above that voters in regional contests choose solely according to spatial proximity. In turn, by the single-peakedness of spatial preferences we then have the following Lemma (trivial proof omitted):

Lemma 1 (Regional Elections)

- *If* $\left|x_m^R - x_P^R\right| < \left|x_m^R - x_{\sim P}^R\right|$ *then* P *wins the regional election in* R.
- *If* $x_A^R = x_B^R$ *then all voters in* R *randomize and* $\pi_A^R(\mathbf{x}) = \pi_B^R(\mathbf{x}) = \frac{1}{2}$.
- *If* $x_A^R \neq x_B^R$ *but* $\left|x_m^R - x_A^R\right| = \left|x_m^R - x_B^R\right|$, *then the median voter in* R *randomizes and* $\pi_A^R(\mathbf{x}) = \pi_B^R(\mathbf{x}) = \frac{1}{2}$.

Whichever party's local platform x_P^R is closer to the local median x_m^R wins the regional election, except when indifference conditions leave each party with a 50% chance of winning. Note that this does not imply that regional candidates actually *choose* the regional median; indeed, they will often deviate from their regional median to help their party in national elections. To study this dynamic, we need to understand how national election vote shares react to regional platforms.

For the remainder of the paper we assume that $x_A^N \leq 1 \leq x_B^N$, i.e. that A is a 'left' party and B is a 'right' party, and that both can only announce national platforms on their respective sides of the spectrum, up and including the national median $x_m^N = 1$. While simplifying the proofs considerably, this assumption is not necessary for the paper's results. The following Lemma tells us that, despite a more complex utility function, voter choice in the national election *within a particular region* still satisfies the single-crossing property. The proof is in Appendix A:

Lemma 2 *For any voter* x_i^R*:*

- *If* $u_i^A(\mathbf{x}) > u_i^B(\mathbf{x})$ *then the same is true for any voter* j *such* $x_j^R < x_i^R$.
- *If* $u_i^B(\mathbf{x}) > u_i^A(\mathbf{x})$ *then the same is true for any voter* j *such* $x_j^R > x_i^R$.

Lemma 2 allows us to identify a *swing voter* x_s^R in R who is perfectly indifferent between parties A and B in the national election (proof in Appendix A):

$$x_s^R(\mathbf{x}) = \{x_i : u_i^A(\mathbf{x}) = u_i^B(\mathbf{x})\} \Rightarrow$$
$$x_s^R(\mathbf{x}) = \underbrace{\left(\frac{x_A^N + x_B^N}{2}\right)}_{Spatial\ Midpoint} + \underbrace{\left(\frac{k\left(|x_B^R - x_B^N| - |x_A^R - x_A^N|\right) + \sigma}{2(x_B^N - x_A^N)}\right)}_{Party\ Unity}. \quad (5)$$

Since $x_A^N \leq 1 \leq x_B^N$, all voters with ideal points higher (lower) than x_s^R choose B (A). The first term in (5) is the spatial midpoint between the two national platforms. As that mid-point moves higher, more voters choose party A. The second term captures both the party unity effect, as well as the effect of the exogenous disturbance σ. All else constant, as disunity increases in party B, the position of $x_s^R(\mathbf{x})$ increases, meaning more voters choose A. The opposite occurs when disunity increases in party A. Finally, $x_s^R(\mathbf{x})$ increases in the realization of σ. Looking at the denominator of the second term, we see that, as the two parties' national platforms converge, the second term crowds out the first term in determining the position of the swing voter: since parties are less differentiated, voter choice is entirely determined by party unity and the exogenous shock σ.

Let v_P^R be the proportion of voters from region R who choose P in the national election. Given our distributional assumptions and the simplifying assumption that $x_A^N < 1 < x_B^N$, we can write v_A^R as:[3]

$$v_A^R(\mathbf{x}) = \begin{cases} x_s^R(\mathbf{x}) - b_R & \text{if} \quad x_s^R(\mathbf{x}) \in [b_R, 1 + b_R] \\ 0 & \text{if} \quad x_s^R(\mathbf{x}) < b_R \\ 1 & \text{if} \quad x_s^R(\mathbf{x}) > 1 + b_R \end{cases} . \quad (6)$$

Panel (b) in Figure 1 above provides a visualization of the parties' national election vote shares in region 1.

In turn, party A's aggregate vote share is $v_A^N(\mathbf{x}) = \left(\frac{v_A^1(\mathbf{x}) + v_A^2(\mathbf{x})}{2}\right)$, and $v_B^N(\mathbf{x}) = 1 - v_A^N(\mathbf{x})$. Given that all contests are plurality rule, we can write the probability

[3]If $x_A^N = x_B^N = 1$ then either: a) the party with the most attractive combination of organizational unity and valence wins the entire vote in region R, i.e. $v_P^R = 1$; or b) organizational disunity and valance are such that voters in R are indifferent between the two national parties and randomize.

that P wins the national contest $\pi_P^N(\mathbf{x})$ as follows:

$$\pi_P^N(\mathbf{x}) = \text{Prob}\left[v_P^N(\mathbf{x}) > \frac{1}{2}\right]. \tag{7}$$

From (6) above, in the case of Party A this is written:

$$\pi_A^N(\mathbf{x}) \;=\; \text{Prob}\left[v_A^N(\mathbf{x}) > \frac{1}{2}\right] \;=\; \text{Prob}\left[\frac{x_s^1(\mathbf{x}) + x_s^2(\mathbf{x}) - b_1 - b_2}{2} > \frac{1}{2}\right]. \tag{8}$$

In turn, using (5) above and the fact that $b_1 = 2 - b_2$, we can substitute and re-arrange the right-hand-side of (8) to arrive at the following expression:

$$\pi_A^N(\mathbf{x}) = \text{Prob}\left[\sigma > (x_B^N - x_A^N)(2 - x_B^N - x_A^N) - \frac{k(\Delta_B - \Delta_A)}{2}\right], \tag{9}$$

where $\Delta_P \equiv \left(\left|x_P^1 - x_P^N\right| + \left|x_P^2 - x_P^N\right|\right)$ represents the total disunity in party P. Recalling that σ is uniformly distributed over $\left[-1, 1\right]$, and using the fact that $(x_B^N - x_A^N)(2 - x_B^N - x_A^N) = 2(x_B^N - x_A^N) - (x_B^N - x_A^N)(x_B^N + x_A^N)$ we rewrite (9) as:

$$\pi_A^N(\mathbf{x}) = \frac{1}{2} \underbrace{- (x_B^N - x_A^N) + \frac{(x_B^N - x_A^N)(x_B^N + x_A^N)}{2}}_{\text{spatial proximity}} + \underbrace{\frac{k(\Delta_B - \Delta_A)}{4}}_{\text{party unity}}. \tag{10}$$

Naturally $\pi_B^N(\mathbf{x}) = 1 - \pi_A^N(\mathbf{x})$.[4] This expression captures how expected vote shares respond to considerations of both spatial proximity and party unity. The spatial proximity term tells us that, holding party unity constant, A^N's probability of winning the national election increases as x_A^N becomes closer to the national median at $x_m^N = 1$ and as x_B^N moves further from the national median.[5] The party unity term tells us that, holding proximity to the national median constant, A^N's probability of winning the national election increases as Δ_B increases and as Δ_A decreases. Of course, considerations of spatial moderation and organizational unity cannot be studied in isolation: platform choices designed to increase proximity to

[4] If $x_B^N = x_A^N$ and $\Delta_A = \Delta_B$, each party wins the national contest with $\pi_P^N(\mathbf{x}) = \frac{1}{2}$.

[5] The derivative of the spatial proximity term with respect to x_A^N is $1 - x_A^N$, which is (weakly) greater than 0 since $x_A^N \leq 1$ by assumption.

the national median will have consequences for party unity, and platform choices designed to increase party unity will impact one's standing with the national median. The following sections examine how candidates negotiate these tradeoffs under different exogenous scenarios.

3.1 The Equilibrium Possibility Space

Our solution concept is pure strategy Nash Equilibrium, simply referred to as 'equilibrium'. As with most similar models, we do not examine mixed platform strategies. Since the game has multiple candidates competing in multiple elections, it yields a highly complex action space. Thankfully, the following Theorem significantly narrows the set of possible equilibria:

Theorem 1 (Convergence) *In any pure strategy Nash Equilibrium one of two convergence patterns must emerge:*

- *Median Convergence:* $x_P^R = x_m^R$ *for all four regional candidates and* $x_A^N = x_B^N = 1$.
- *Partisan Convergence:* $x_A^N = x_A^1 = x_A^2$ *and* $x_B^N = x_B^1 = x_B^2$.

This Theorem reduces the set of possible equilibria with disunified parties to 1, and demonstrates that in this disunified equilibrium, national platforms converge to the national median. Henceforth, we label this median convergence strategy vector as the *Geographic Balancing* strategy vector, denoted $\mathbf{x^G}$. While $\mathbf{x^G}$ is the only possible equilibrium with heterogeneous parties, there is no *a priori* limitation to the range of equilibria which meet the second convergence criterion in Theorem 1, which we label *Party Unity* strategy vectors. Indeed, we show below that when the value of β is sufficiently large the game will be characterized by a multiplicity of equilibria with perfectly unified organizations. In such cases, though our model does not make a unique prediction, we derive the range of platforms which would be allowable in equilibrium.

The proof of Theorem 1 (Appendix A.1) is grounded in Lemma A2, which establishes that, in any situation where regional candidates in R announce distinct platforms ($x_A^R \neq x_B^R$), those platforms must be fully convergent with their party's national platform ($x_P^R = x_P^N$). If this is not the case, one of three profitable deviations exists: a) a regional candidate can improve their chances of winning the national contest without impacting their chances in the regional contest; b) a regional candidate can improve their chances of winning the regional contest without impacting their chances in the national contest; or c) a national candidate can increase their utility by deviating to the national median. In one brushstroke, this Theorem drastically reduces the number of strategy vectors we must consider in characterizing the game's equilibria, and thus makes solving the model feasible. The Conclusion discusses additional applications of this basic result.

Proposition 1 identifies the conditions under which Geographic Balancing is an equilibrium (proof in Appendix A.2).

Proposition 1 (Balancing) *If* $\beta \leq \frac{4}{kd}$ *then* $\mathbf{x^G}$ *is a Nash Equilibrium.*

At $\mathbf{x^G}$ all three elections are decided by an unbiased coin-flip: all candidates adopt identical positions in their respective contests, and parties are equally disunified ($\Delta_A = \Delta_B$). In turn, regional candidates receive an expected payoff of $U_P^R = \left(\frac{1+\beta}{2}\right)$, while national candidates receive $U_P^N = \frac{1}{2}$. Note that national candidates A^N and B^N have no incentive to deviate. Firstly, holding party unity constant, such a deviation moves the candidate away from the national median. Secondly, this spatial cost cannot be compensated for by increasing party unity: since P's two regional politicians are on opposite sides of the ideological spectrum (one at x_m^1 and the other at x_m^2), any platform shift towards one of the regional medians increases unity in one region but *creates* greater disunity in the other. Hence the notion of 'balancing': when regional candidates are anchored at the regional median, national candidates become anchored at the national median.

Moving to regional candidates, note via Lemma 1 that any deviation by a regional candidate from $\mathbf{x^G}$ causes them to lose their regional election. It follows that their optimal deviation is to $x_P^N = x_m^N = 1$, which minimizes their organizational disunity and maximizes their chances in the national contest. It is straight-forward to show that their new probability of winning the national contest with this deviation is $\pi_A^N(\cdot) = \frac{1}{2} + \frac{kd}{4}$. By comparing payoffs and re-arranging we derive the condition $\beta \leq \frac{4}{kd}$ in Proposition 1.

Regarding comparative statics, Geographic Balancing is more likely to occur when both the baseline polarization d and the disunity penalty k are low. High d means that, when regional candidates deviate from $\mathbf{x^G}$ to $x_m^N = 1$, they create a large increment of additional unity in their party. This in turn enhances the impact of their deviation on $\pi_P^N(\cdot)$, making said deviation more appealing. The same reasoning applies to high k. Secondly, it is more likely to occur in decentralized systems where β is low. In these systems, regional candidates will be less inclined to forgo their regional election in order to create greater party unity. Decentralization thus has the ability to incentivize moderation at the national-level *even if* the electorate is highly polarized (d is high) and is highly sensitive to internal disunity (k is high). This combination of heterogeneous but nationally moderate parties may present an attractive option for states with strong territorial cleavages.

4 Polarization and Party Unity

We now turn to the second equilibrium format identified in Theorem 1, in which all candidates from P choose identical platforms for both $P \in \{A, B\}$. The Geographic Balancing equilibrium identified in Proposition 1 minimizes political polarization at the national level, since it involves maximally centrist national-level platforms ($x_A^N = x_B^N = 1$). On the other hand, the strategy vector with unified parties located at the national median $x_m^N = 1$ is never an equilibrium.

Remark 1 *The strategy vector at which $x_A^1 = x_A^2 = x_A^N = x_m^N$ and $x_B^1 = x_B^2 = x_B^N = x_m^N$ is not a Nash Equilibrium.*

To see this, note that when parties are perfectly unified at the national median, all four regional candidates win their regional contest with probability $\pi_P^R = \frac{1}{2}$. In turn, any of the regional candidates could make a tiny platform deviation towards their region's median, and win their local contest with certainty while only infinitesimally impacting their party's probability of winning national election. Equilibria with unified parties must thus exhibit some level of divergence in parties' national platforms. The core questions become exactly how *polarized*, and what *form* this polarization assumes. The following Lemma begins to answer that question (proof in Appendix B).

Lemma 3 (National Candidates) *In any Party Unity Nash Equilibrium, it must be the case that* $x_A^N \geq 1 - \frac{k}{2}$ *and* $x_B^N \leq 1 + \frac{k}{2}$.

Define $\underline{x}_k \equiv 1 - \frac{k}{2}$ and $\overline{x}_k \equiv 1 + \frac{k}{2}$, and note that $\underline{x}_k$ and $\overline{x}_k$ are positioned symmetrically around the national median ($x_m^N = 1$). Lemma 3 thus provides a symmetric 'ceiling' on the extent of polarization possible in any Party Unity equilibrium. These conditions follow from the deviation incentives of national candidates. Consider candidate A^N. Given any strategy vector at which both parties are perfectly unified, she must decide whether to make an incremental move towards $x_m^N = 1$, a deviation which improves her ideological standing with the national median voter but creates additional organizational disunity. As demonstrated in the proof, for any $x_A^N < \underline{x}_k$ the value of this deviation will outweigh the costs in terms of additional disunity. This is a result of the quadratic shape of voter's loss function for spatial proximity, meaning that the more extreme is x_A^N, the greater the impact of an incremental centripetal deviation, and the more tempting such a deviation will be. In contrast, for any $x_A^N \geq \underline{x}_k$ the impact of this deviation drops and A^N prefers not to create the additional disunity. As k increases, so does the marginal cost of creating party disunity. As a result, as k increases so does the symmetric range of possible Party Unity equilibria $[\underline{x}_k, \overline{x}_k]$.

Lemma 3 identifies a necessary but not sufficient condition for Party Unity equilibrium: it identifies the range of Party Unity strategy vectors from which A^N and B^N will not deviate. To fully characterize Party Unity equilibria, we must also examine regional candidates' deviation incentives. Indeed, from Remark 1 we know that there is at least one Party Unity strategy vector in the range $[\underline{x}_k, \overline{x}_k]$ from which regional candidates will *always* deviate.

As it turns out, there are three key Party Unity strategy vectors which must be assessed to characterize the set of possible Party Unity equilibria. The first is *Regional Medians*, labeled $\mathbf{x^R}$, at which both parties are unified at the respective regional median positions ($x_A^N = x_A^1 = x_A^2 = x_m^1$ and $x_B^N = x_B^1 = x_B^2 = x_m^2$). The second is *Symmetric Polarization*, labeled $\mathbf{x^S}$, at which both parties are unified at the most extreme platforms allowable by Lemma 3 ($x_A^N = x_A^1 = x_A^2 = \underline{x}_k$ and $x_B^N = x_B^1 = x_B^2 = \overline{x}_k$). The third are the *Asymmetric Polarization* vectors at which one party is unified at the national median and the other at their regional extreme, of which there are two: $\mathbf{x^{\underline{Y}}}$ where A is extreme and B is moderate ($x_A^N = x_A^1 = x_A^2 = \underline{x}_k$ and $x_B^N = x_B^1 = x_B^2 = 1$), and $\mathbf{x^{\overline{Y}}}$ where B is extreme and A is moderate ($x_A^N = x_A^1 = x_A^2 = 1$ and $x_B^N = x_B^1 = x_B^2 = \overline{x}_k$).

At the strategy vectors $\mathbf{x^R}$ and $\mathbf{x^S}$, the winners of the regional elections are A^1 and B^2. We refer to any strategy vector with this regional win profile as a 'Type 1' strategy vector. On the other hand, under the right conditions (see below), at the strategy vectors $\mathbf{x^{\underline{Y}}}$ and $\mathbf{x^{\overline{Y}}}$ it is possible that regional candidates from the same party win both regional contests. We refer to any strategy vector with this regional win profile as a 'Type 2' strategy vector. As demonstrated below, from among all Type 1 strategy vectors, either $\mathbf{x^R}$ or $\mathbf{x^S}$ is pivotal in determining the game's equilibrium profile; and from among all Type 2 strategy vectors, $\mathbf{x^{\underline{Y}/\overline{Y}}}$ are pivotal in determining the game's equilibrium profile. Thus, while these strategy vectors do not exhaust the set of possible equilibria, they provide theoretical thresholds which allow us fully characterize the game's solution.

4.1 Regional Medians and Symmetric Polarization

Under what conditions will both parties unify at the respective regional medians? Begin by considering national candidates' decision. From Lemma 3 we know that A^N (B^N) will not deviate from $\mathbf{x^R}$ as long as $x_m^1 \geq \underline{x}_k$ ($x_m^2 \leq \overline{x}_k$). Note that, by simple substitution, the conditions $x_m^1 \geq \underline{x}_k$ and $x_m^2 \leq \overline{x}_k$ can both be rewritten as $d \leq k$, which follows form the fact that $\underline{x}_k$ and $\overline{x}_k$ are positioned symmetrically around $x_m^N = 1$. In contrast, if $d > k$ then via Lemma 3 candidates A^N and B^N will have a profitable deviation from $\mathbf{x^R}$.

Moving to regional candidates, since $\mathbf{x^R}$ is a Type 1 strategy vector, neither candidate A^1 nor B^2 has any incentive to deviate: they are already winning the regional election, and have no incentive to create greater intra-party disunity. On the other hand, at $\mathbf{x^R}$ candidates A^2 and B^1 are 'pulled' to the ideal point of the regional median on the opposite side of the political spectrum, and thus lose their regional contests. Proposition 2 identifies the condition under which these two losing regional candidates can be induced to keep their parties unified rather than pursue local incumbency (proof in Appendix B.1):

Proposition 2 (Regional Medians) *If $d \leq k$ and $\beta \geq \frac{2}{kd}$ then $\mathbf{x^R}$ is an equilibrium. If either condition is not met, $\mathbf{x^R}$ is not an equilibrium.*

Thus, both $d \leq k$ and $\beta \geq \frac{2}{kd}$ are necessary conditions, and they are jointly sufficient. If $d \leq k$ then national candidates do not deviate from $\mathbf{x^R}$, and we need to check the deviation conditions for candidates A^2 and B^1. These two candidates have only one possible profitable deviation: switching from the opposing region's median position to their own region's median, giving them a $\pi_P^R(\cdot) = \frac{1}{2}$ chance of winning their regional contest. However, this creates additional disunity in their party, thus reducing its national election probability $\pi_P^N(\cdot)$. As demonstrated in the proof, this deviation is not profitable as long as $\beta \geq \frac{2}{kd}$.

This condition is easier to satisfy when β is large: when regional candidates place a high value on their party holding national office they are less likely to deviate and create additional party disunity. This willingness to forgo local incumbency is accentuated when k is large, since large k increases the costs of creating greater disunity in their party. The impact of baseline polarization d is more complex. On the one hand, increasing d makes the condition $d \leq k$ harder to satisfy, making it *more* likely that national candidates will deviate from $\mathbf{x^R}$. On the other, increasing

d makes the condition $\beta \geq \frac{2}{kd}$ easier to satisfy, making it *less* likely that A^2 and B^1 deviate from $\mathbf{x^R}$: when d increases, so does the level of disunity created by deviating to one's own regional median. As a result, it is impossible to sustain $\mathbf{x^R}$ at very low *and* at very high values of d. Rather, $\mathbf{x^R}$ emerges as an equilibrium at *intermediate* values of baseline polarization. We develop the full implications of this non-monotonicity below.

Lemma A5 in Appendix B.1.1 demonstrates the relevance of $\mathbf{x^R}$ in characterizing the game's general solution. In particular, it demonstrates that, for the case in which $d \leq k$, the Regional Medians strategy vector $\mathbf{x^R}$ is the *easiest* Type 1 Party Unity strategy vector to sustain in equilibrium, i.e. it is sustainable for a wider range of β than any other Type 1 strategy vector. In turn, for the case in which $d \leq k$, the value $\beta = \frac{2}{kd}$ becomes an important theoretical threshold, below which no Type 1 strategy vector can be sustained in equilibrium.

As was the case with $\mathbf{x^R}$, at the Symmetric Polarization strategy vector $\mathbf{x^S}$, parties are unified and symmetrically positioned on either side of the national median $x_m^N = 1$. The distance between the two parties' national platforms at $\mathbf{x^R}$ is $(x_m^2 - x_m^1) = d$ and at $\mathbf{x^S}$ is $(\underline{x}_k - \overline{x}_k) = k$. In turn, the polarization of national platforms $(x_B^N - x_A^N)$ is lower at $\mathbf{x^S}$ than at $\mathbf{x^R}$ as long as $d > k$, while it is (weakly) higher at $\mathbf{x^S}$ than at $\mathbf{x^R}$ as long as $d \leq k$. From Lemma 3 we know that neither of the national candidates have a profitable deviation from $\mathbf{x^S}$. Like national candidates, neither A^1 nor B^2 has any incentive to deviate, as they win their respective regional contests. The following Proposition demonstrates that, if β is large enough, the same will be true of candidates A^2 and B^1 (proof in Appendix B.1). In these circumstances, political parties' desire for party unity allows them to become 'trapped' at the political extremes.

Proposition 3 (Symmetric Polarization) *If* $\beta \geq \max\{\frac{4}{k^2}, \frac{4}{kd}\}$ *then* $\mathbf{x^S}$ *is an equilibrium; if not then* $\mathbf{x^S}$ *is not an equilibrium.*

The first thing to note is that Symmetric Polarization requires high β. At first glance, one might suppose that the dependence of regional candidates on national office would push equilibrium policies towards the national median. In fact, the opposite happens. When β is large, candidates A^2 and B^1 are incentivized to maintain party unity, even if this means adopting platforms on the opposite side of the political spectrum. Symmetric Polarization also requires high d. To see this, note that when $d \leq k$ the relevant condition from Proposition 3 is $\beta > \frac{4}{kd}$, and that this condition becomes easier to satisfy as d increases. The mechanism is once again that raising d increases the size of the deviation that losing regional candidates must make in order to win their regional contest, this making said deviations more costly.[6] Finally, and for the same reasons, increasing k makes $\mathbf{x^S}$ easier to sustain, as this amplifies the cost of creating party disunity.[7]

It is interesting to compare Propositions 2 and 3. The value of $\max\{\frac{4}{k^2}, \frac{4}{kd}\}$ from Proposition 3 depends on the relative size of d and k. In particular, if $d > k$ then $\max\{\frac{4}{k^2}, \frac{4}{kd}\} = \frac{4}{k^2}$, while if $d \leq k$ then $\max\{\frac{4}{k^2}, \frac{4}{kd}\} = \frac{4}{kd}$. Note that the condition $d \leq k$ is precisely that which defined the possibility of a Regional Medians equilibrium in Proposition 2. Lemma A4 in Appendix B.1.1 demonstrates that, for the case $d > k$, where $\mathbf{x^R}$ is not an equilibrium, $\mathbf{x^S}$ is the *easiest* Type 1 Party Unity strategy vector to sustain in equilibrium. When $d > k$, the value $\beta = \frac{4}{k^2}$ thus represents an important theoretical threshold, below which no Type 1 strategy vector can be sustained in equilibrium.

It is also interesting to compare Propositions 1 and 3. When $d \leq k$, the relevant threshold from Proposition 3 is $\frac{4}{kd}$. This, in turn, is the same as the threshold identified in Proposition 1, with the only difference being the direction of the inequality. Similarly, when $d > k$, the relevant threshold from Proposition 3 is $\frac{4}{k^2}$, and since $d > k$ the relevant threshold from Proposition 1 is lower than that from Proposition 3 ($\frac{4}{kd} < \frac{4}{k^2}$). In words, Geographic Balancing $\mathbf{x^G}$ and Symmetric Polarization

[6]As d reaches k and then moves to values $d > k$, the relevant condition becomes $\frac{4}{k^2}$, and further increases in d no longer impact likelihood that $\mathbf{x^S}$ is an equilibrium.

[7]Corollary A2 in Appendix B.1 demonstrates that, when $d \leq k$, the result in Proposition 3 actually applies not only to $\mathbf{x^S}$ but also to any Party Unity strategy vector that meets two criteria: a) the two parties are symmetrically located around the national median $x_m^N = 1$ ($x_A^N = 2 - x_B^N$); and b) both parties are more radical than their regional medians ($x_A^N < x_m^1$ and $x_B^N > x_m^2$). We exhibit this feature of the case $d \leq k$ in our study of comparative statics below (Section 5).

$\mathbf{x}^{\mathbf{S}}$ are *mutually exclusive*,[8] and the exogenous conditions which facilitate one are precisely those that undermine the other.

4.2 Asymmetric Polarization

Propositions 2 and 3 apply to Type 1 strategy vectors $\mathbf{x}^{\mathbf{R}}$ and $\mathbf{x}^{\mathbf{S}}$, at which A^1 and B^2 win their respective regional elections. We now move to Party Unity strategy vectors in which one of the two parties wins both regional contests, labeled Type 2 strategy vectors. Without loss of generality, focus on the case in which B^1 wins in region 1 and B^2 wins in region 2. Lemma A7 in Appendix B.2 begins by establishing that, unless $d < \frac{k}{2}$, no Type 2 strategy vector can be an equilibrium. If this condition is not met, then at any Party Unity strategy vector where B^1 defeats A^1 in region 1, candidate A^N has a profitable deviation.

Consider the Asymmetric Polarization strategy vector $\mathbf{x}^{\underline{\mathbf{Y}}}$, at which A is unified at $x_A^N = \underline{x}_k$ and B is unified at $x_B^N = 1$. Via symmetry the following Proposition (proof in Appendix B.2) applies equally to $\mathbf{x}^{\overline{\mathbf{Y}}}$.

Proposition 4 *For the case in which $d < \frac{k}{2}$, the Asymmetric Polarization strategy vector $\mathbf{x}^{\underline{\mathbf{Y}}}$ is a Nash Equilibria if and only if $\beta \geq \frac{8}{k(k-2d)}$.*

When $d < \frac{k}{2}$, at $\mathbf{x}^{\underline{\mathbf{Y}}}$ party B's candidates win both regional elections, and have no deviation incentive. Furthermore, from Lemma 3 we know that national candidates have no incentive to deviate from $\mathbf{x}^{\underline{\mathbf{Y}}}$. In this case, the equilibrium relevant candidate is A^1 (Lemma A8 in Appendix B.2). If they deviate to a platform just close enough to x_m^1 so as to win their regional contest, they also create greater disunity in their party, and thus reduce their chances of winning the national contest. As long as long as $\beta \geq \frac{8}{k(k-2d)}$ such a deviation is not profitable.

Like Symmetric Polarization $\mathbf{x}^{\mathbf{S}}$, Asymmetric Polarization $\mathbf{x}^{\underline{\mathbf{Y}}}$ becomes easier to sustain at high β and high k: as both increase, so does the cost that A^1 pays for creating greater disunity in her party. On the other hand, the impact of d

[8] Technically, in the limiting case where $\beta = \frac{4}{kd}$ and $d \leq k$, both $\mathbf{x}^{\mathbf{G}}$ and $\mathbf{x}^{\mathbf{S}}$ are equilibria in *weakly-dominant* strategies.

on Asymmetric Polarization is markedly different from its impact on Symmetric Polarization. While the latter is more likely to be supported when d is large, Asymmetric Polarization is more likely to be supported when d is small – the smaller is d, the easier to satisfy $\beta \geq \frac{8}{k(k-2d)}$. If d is small, then $x_B^1 = 1$ is much closer to x_m^1 than is $x_A^1 = \underline{x}_k$. In turn, the deviation required by candidate A^1 to win her regional election is larger, and causes greater damage to her party's unity. Put simply, *if a system is sufficiently centralized, Asymmetric Polarization can exist* ***even if*** *the electorate itself is moderate.*

Lemma A9 in Appendix B.2 demonstrates that, for for the case in which $d < \frac{k}{2}$, the Asymmetric Polarization strategy vectors $\mathbf{x}^{\underline{\mathbf{Y}},\overline{\mathbf{Y}}}$ are the *easiest* Type 2 Party Unity strategy vectors to sustain in equilibrium. In turn, for the case in which $d < \frac{k}{2}$, the value $\beta = \frac{8}{k(k-2d)}$ becomes an important theoretical threshold, below which no Type 2 strategy vector can be sustained in equilibrium.

5 Political Geography and Polarization

We now examine the relationship between the game's exogenous parameters and national-level polarization, defined as the equilibrium distance between national platforms $(x_B^N - x_A^N)$. We first identify a small portion of the parameter space in which the game has no pure strategy equilibrium (proof in Appendix C).

Corollary 1 (Non-Existence) *The game has no pure strategy Nash Equilibrium when the following two conditions are both met:* $d > k$ *and* $\frac{4}{dk} < \beta < \frac{4}{k^2}$.

Conversely, if either of these conditions is violated, then by Propositions 1-4 the game has at least one pure strategy equilibrium. This Corollary applies to a narrow set of circumstances in which baseline polarization d is large enough to ensure that national candidates deviate from Regional Medians; while β adopts intermediate values such that at least one regional candidate deviates from Geographic Balancing and Asymmetric Polarization. This small range of indeterminacy does not undermine our ability to draw clear theoretical conclusions.

As highlighted above, large β puts a premium on winning the national contest, which in turn puts a premium on party unity. As a result, parties may become trapped at the regional extremes. We now develop this insight more completely. In Figure 2 on the following page, the x-axes represent β and the y-axes represent equilibrium platform polarization $(x_B^N - x_A^N)$. The cells below the x-axes list the equilibria that can be sustained for that specific range of β. The red-lined plots above the x-axis capture the levels of platform polarization that occur in equilibrium. Note that, since we do not place a ceiling on β, the right-side of all lines in these plots have arrows. To facilitate presentation, all three plots set $k = 1$, but differ in their levels of baseline polarization d: the top plot applies to a polarized electorate ($d = \frac{4}{5}$), the second to a moderately polarized electorate ($d = \frac{1}{2}$), and the bottom to a non-polarized electorate ($d = \frac{1}{10}$).

By choosing $k = 1$, we ensure that the condition $d \leq k$ is satisfied. In turn, from Propositions 1-3 this implies: a) Regional Medians $\mathbf{x^R}$ is an equilibrium if $\beta \geq \frac{2}{kd}$, and b) the value $\beta = \frac{4}{kd}$ represents a threshold above which Geographic Balancing disappears as an equilibrium while Symmetric Polarization appears.[9] Begin with the top plot, where baseline polarization is $d = \frac{4}{5}$. In this case, the condition $d < \frac{k}{2}$ is not met, meaning that we do not need to consider Type 2 strategy vectors when considering equilibrium thresholds. In turn, the thresholds that become relevant are those from Propositions 1, 2, and 3. At low values of $\beta < \frac{2}{kd} = 2.5$, the only possible equilibrium outcome is Geographic Balancing $\mathbf{x^G}$, at which national-level platform polarization is $(x_B^N - x_A^N) = 0$. In the range $2.5 \leq \beta \leq \frac{4}{kd} = 5$ both Geographic Balancing and Regional Medians $\mathbf{x^R}$ can be sustained (hence two red-lines), and the Regional Medians outcome yields polarization $(x_B^N - x_A^N) = d = \frac{4}{5}$.

Finally, for $\beta > 5$ Geographic Balancing is replaced by Symmetric Polarization at which $(x_B^N - x_A^N) = (\bar{x}_k - \underline{x}_k) = 1$. Since $d \leq k$, above the threshold $\beta = 5$ there exists a multiplicity of possible polarized equilibria, one of which is $\mathbf{x^S}$ (Corollary A2, Appendix B.1), captured by the shaded space between the downward sloping

[9]From Proposition 3, we know that when $d \leq k$, $\mathbf{x^S}$ is an equilibrium if $\beta \geq \frac{4}{kd}$. From Proposition 3, we know that $\mathbf{x^G}$ is an equilibrium if $\beta \leq \frac{4}{kd}$.

Figure 2: Platform Polarization and β

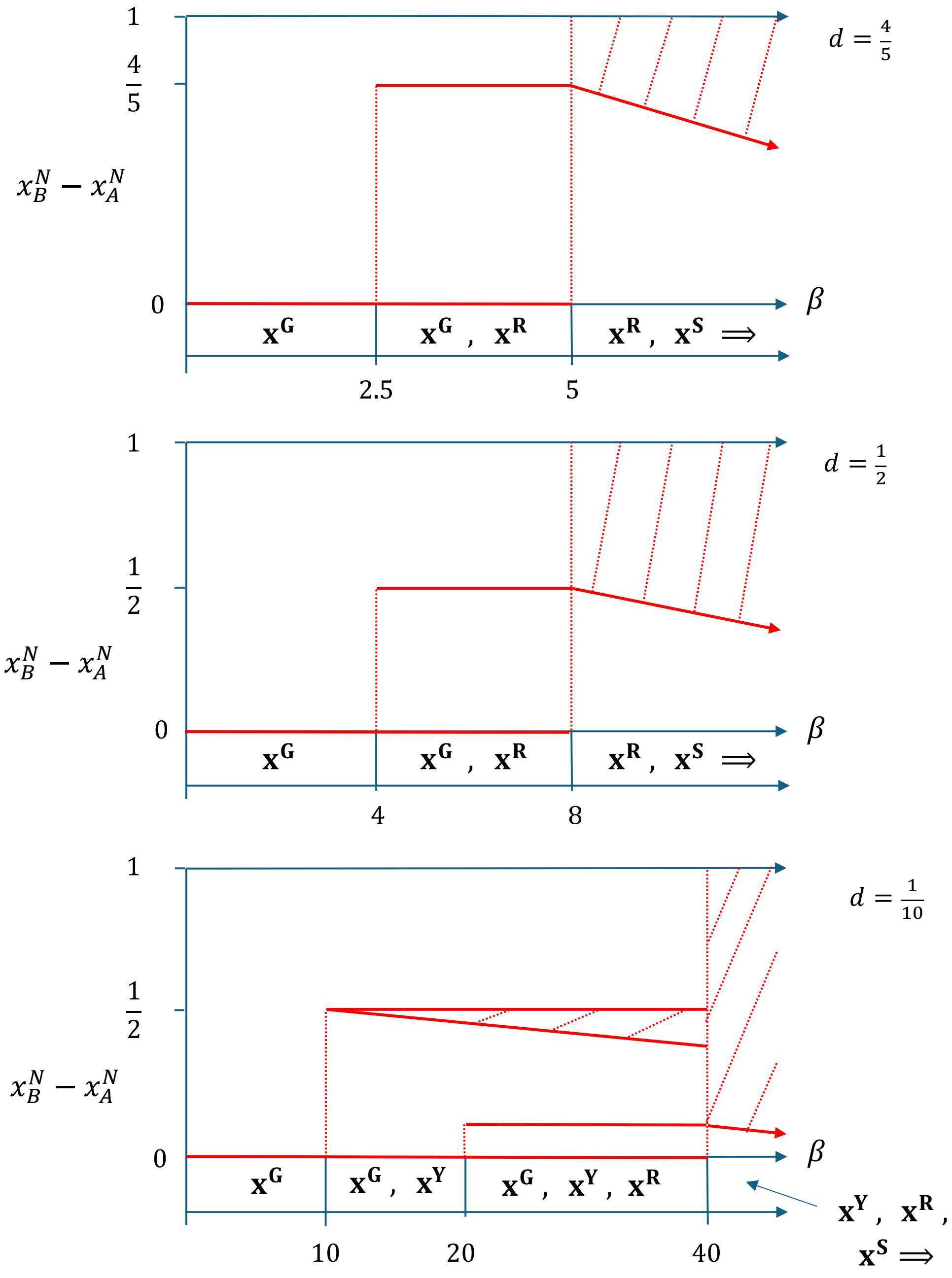


Figure 2: Since $k = 1$, by Lemma 3 the maximum level of polarization possible is $(\bar{x}_k - \underline{x}_k = 1)$. For ease of presentation, the scale of the x-axis changes from plot to plot.

red-arrow and the ceiling condition at $(x_B^N - x_A^N) = 1$. This arrow is downward sloping because as β increases so does the range of possible Party Unity equilibria (Lemma A6, Appendix B.1.1). Indeed, as β becomes arbitrarily large, so does the set of Party Unity equilibria (subject to the constraint of Lemma 3 above).

The middle plot in Figure 2 reduces baseline polarization to the intermediate level $d = \frac{1}{2}$. As compared to the top plot Geographic Balancing exists as a unique equilibrium for a wider range of $\beta < 4$. Furthermore, when Regional Medians does emerge as an equilibrium for $\beta \geq 4$, it implies a lower level of polarization than in the top plot (the distance between the regional medians is $d = \frac{1}{2}$). Finally, note that Symmetric Polarization only emerges at the values $\beta \geq 8$, and that when it does emerge the range of possible equilibria is higher than and more moderate than in the top plot.

The bottom plot in Figure 2 sets $d = \frac{1}{10}$, which means that $d < \frac{k}{2}$ and Asymmetric Polarization becomes relevant in establishing equilibrium thresholds. Indeed, when $d = \frac{1}{10}$ and $k = 1$ it becomes much easier to sustain Asymmetric Polarization than it is to support $\mathbf{x^R}$ or $\mathbf{x^S}$.[10] Once again, at low values of $\beta < 10$ only Geographic Balancing is possible. For $\beta \geq 10$ Asymmetric Polarization becomes possible, and since $k = 1$ platform polarization is $(1 - \underline{x}_k = \frac{k}{2})$. Finally, Regional Medians and Symmetric Polarization only emerge at higher values of β.

5.1 Geographic vs. Systemic Polarization

We now move to the relationship between baseline polarization $d = (x_m^2 - x_m^1)$ and equilibrium platform polarization $(x_B^N - x_A^N)$. To do so, we rewrite the threshold results from Propositions 1-4 to isolate d. The threshold from Proposition 1 ($\beta \leq \frac{4}{kd}$) becomes $d \leq \frac{4}{k\beta}$; the threshold from Proposition 2 ($\beta \geq \frac{2}{kd}$) becomes $d \geq \frac{2}{k\beta}$; the threshold from Proposition 3 ($\beta \geq \frac{4}{kd}$) becomes $d \geq \frac{4}{k\beta}$; and the threshold from Proposition 4 ($\beta \geq \frac{8}{k(k-2d)}$) becomes $d \leq \frac{k}{2} - \frac{4}{k\beta}$. From these thresholds emerges the following Corollary (proof in Appendix C):

[10] This is due to the fact, if $d = \frac{1}{10}$ and $k = 1$, then $\frac{8}{k(k-2d)} < \frac{2}{k,d} < \frac{4}{kd}$, i.e. Asymmetric Polarization can be sustained at lower values of β than Symmetric Polarization or Regional Medians.

Corollary 2 (Geographic Balancing) *If $\beta < \frac{2}{k}$, then the only possible equilibrium for all values of d is Geographic Balancing $\mathbf{x^G}$.*

Thus, in sufficiently decentralized systems, changes in baseline polarization d have no impact on equilibrium platform polarization, which is always $(x_B^N - x_A^N) = 0$. Since the unique equilibrium for any political geography is $\mathbf{x^G}$, increasing d makes parties more heterogeneous but no less moderate.

At higher levels of β, the impact of d on platform polarization becomes more pronounced; and unlike the relationship between β and $(x_B^N - x_A^N)$, the relationship between d and $(x_B^N - x_A^N)$ is often non-monotonic, as stated in the following Corollary (Proof in Appendix C).

Corollary 3 (Non-Monotonicity) *As long as one of the following two conditions is met, the relationship between d and $(x_B^N - x_A^N)$ will be non-monotonic:*

- $\frac{8}{k^2} < \beta < \frac{12}{k^2}$,
- $\frac{2}{k} < \beta < \frac{4}{k}$.

The first bullet in Corollary 3 applies to systems with high centralization. As demonstrated in the proof, the fact that $\beta > \frac{8}{k^2}$ ensures that Asymmetric Polarization is possible at low levels of d, while the fact that $\beta < \frac{12}{k^2}$ ensures that β is not inordinately large.[11] In these circumstances the relationship between d and $(x_B^N - x_A^N)$ will be 'U-shaped'. At low values of d Asymmetric Polarization $\mathbf{x}^{\underline{\mathbf{Y}},\overline{\mathbf{Y}}}$ co-exists with Geographic Balancing $\mathbf{x^G}$. As d surpasses $(\frac{k}{2} - \frac{4}{k\beta})$, Asymmetric Polarization disappears as a possibility, leaving a range of d for which only Geographic Balancing exists as an equilibrium. However, the possibility for polarization reappears as d surpasses $\frac{2}{k\beta}$ (when $\mathbf{x^R}$ becomes an equilibrium), and increases again as d surpasses $\frac{4}{k\beta}$ (when $\mathbf{x^S}$ becomes equilibrium).

This pattern emerges in the top plot of Figure 3 (next page), which differs only from Figure 2 only in that d is on x-axis, which is thus bounded by construction between 0 and 1. At very low values of $d < \frac{1}{10}$ both Geographic Balancing and

[11]For example, if $k = \frac{1}{2}$ and $k = \frac{3}{4}$, the constraint $\beta < \frac{12}{k^2}$ requires $\beta < 48$ and $\beta < 21\frac{1}{3}$ respectively.

Figure 3: Platform Polarization and Baseline Polarization

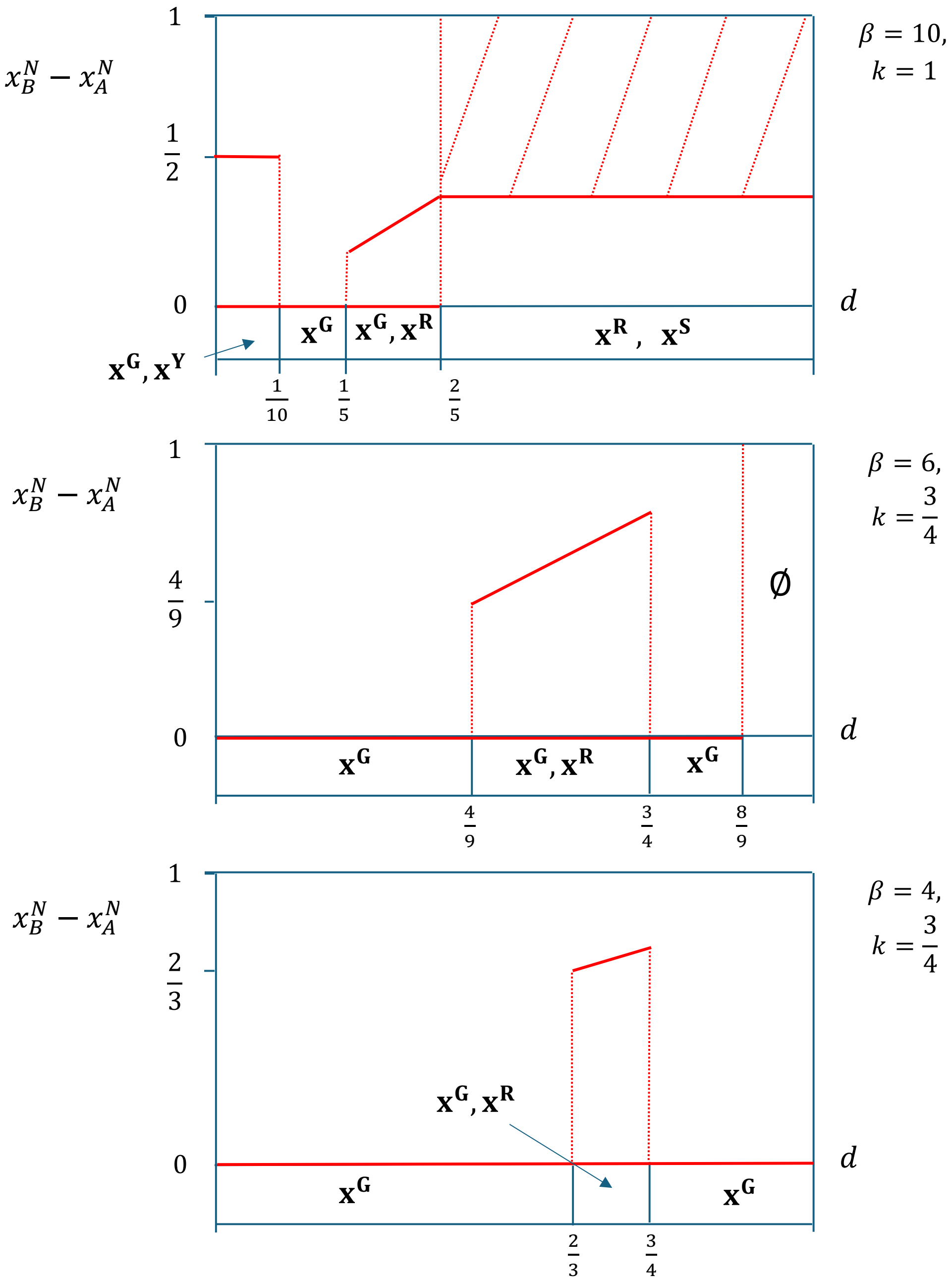


Figure 3: For ease of presentation, in addition to the changing values of β, the value of k switches from $k = 1$ in the top plot to $k = \frac{3}{4}$ in the bottom plots.

Asymmetric Polarization are possible (hence the two red lines). Given that in this Figure $k = 1$ Asymmetric Polarization implies $(x_B^N - x_A^N) = 1 - \underline{x}_k = \frac{1}{2}$. In words, in sufficiently centralized systems, significant polarization can arise (albeit of the Asymmetric variety) despite the fact that x_m^1 and x_m^2 are very close together. Once d crosses $\frac{1}{10}$, Asymmetric Polarization can no longer be supported, leaving only Geographic Balancing in place. From this point on, the relationship between d and $(x_B^N - x_A^N)$ is increasing, as slowly but surely $\mathbf{x^R}$ and then $\mathbf{x^S}$ become possible at higher levels of polarization. As before, for values of $d > \frac{2}{5}$ a range of Symmetric Polarization equilibria become possible.

The second condition in Corollary 3 applies to systems with intermediate centralization. As demonstrated in the proof, the fact that $\beta > \frac{2}{k}$ ensures that Regional Medians $\mathbf{x^R}$ will be an equilibrium at intermediate levels of baseline geographic polarization d. However, the fact that $\beta < \frac{4}{k}$ ensures that, regardless of the size of d, β will not be large enough to sustain either Symmetric or Asymmetric Polarization as an equilibrium. In such circumstances, the relationship between d and $(x_B^N - x_A^N)$ will be 'hump-shaped'. At very low values of $d < \frac{2}{k\beta}$ only Geographic Balancing $\mathbf{x^G}$ will be an equilibrium. Once d surpasses $\frac{2}{k\beta}$, $\mathbf{x^R}$ emerges as a second possibility, thus increasing the possibility for polarization. However, once d surpasses k, $\mathbf{x^R}$ disappears as a possibility, leaving only Geographic Balancing.

This form of non-monotonicity emerges in the bottom plots from Figure 3. At low values of d only Geographic Balancing is possible, meaning national platforms are maximally centripetal $(x_B^N - x_A^N = 0)$. Then, above $d = \frac{2}{k\beta}$ the Regional Medians strategy vector also becomes an equilibrium, adding a fairly polarized equilibrium to Geographic Balancing. Since $(x_B^N - x_A^N) = d$ the level of polarization then increases until the value $d = k$. Above the value $d = k$ Regional Medians can no longer be sustained, and the game drops back to a range of values in which only Geographic Balancing can exist. The key difference between the middle and bottom plots in Figure 3 lies in what happens at high levels of d. In the middle plot $\beta = 6$, and for any $d > \frac{8}{9}$ both of the conditions for Non-Existence identified in Corollary 1 are satisfied ($d > k$ and $\frac{4}{kd} < \beta < \frac{4}{k^2}$) As such, at extremely high values of d the game no longer has a pure strategy equilibrium. In the bottom plot $\beta = 4$, and the condition $\frac{4}{kd} < \beta < \frac{4}{k^2}$ cannot be satisfied for any value of d. As such, at high levels of d the game's unique equilibrium is Geographic Balancing.

These results highlight the complex relationship between polarization in the electorate and the polarization of political parties. Under the right circumstances, an exogenous shift downward in the level of popular polarization may actually lead to an increase in the level of equilibrium platform polarization. For example, for the context captured in Figure 3's top plot, an exogenous shift in geographic polarization from $d = \frac{1}{5}$ to $d = \frac{1}{10}$ could increase equilibrium platform polarization from $(x_B^N - x_A^N) = 0$ to $(x_B^N - x_A^N) = \frac{1}{2}$. Similarly, in the second plot an exogenous shift in geographic polarization from $d = \frac{8}{9}$ to $d = \frac{3}{4}$ could increase equilibrium platform polarization from $(x_B^N - x_A^N) = 0$ to $(x_B^N - x_A) = \frac{3}{4}$.

These non-monotonicities do not exist at all levels of β. For example, in the range $\frac{4}{k^2} < \beta < \frac{8}{k^2}$ the relationship between d and $(x_B^N - x_A^N)$ will be monotonic. At low levels of β only Geographic Balancing will exist. Platform polarization then increases as d increases, first with the addition of a Regional Medians equilibrium, and eventually with the addition of Symmetric Polarization. Thus, the unexpected relationship between d and $(x_B^N - x_A^N)$ does not exist in all institutional environments. However, this does not diminish our point. Corollary 3 identifies a wide range of environments where a shift downward in popular polarization can lead to increased polarization among office-seeking politicians. More generally the model demonstrates that, in sufficiently centralized systems, very low levels of popular polarization can nonetheless co-exist with Asymmetric Polarization in the party system. We find these results relevant against the backdrop of current debates about polarization in contemporary societies.

6 Analytical Implications and Discussion

In the early 2000's the literature on federalism and decentralization took a turn, from welfare economics models of the optimal design of federations to positive analyses of the actual working of federations. The cumulative evidence pointed to a clear message: the classical theory embodied a sort of federal illusion. The idea that decentralization breeds better markets and better democracies, while theoretically cogent, seemed empirically false (Wibbels, 2005; Rodden, 2006; Beramendi, 2012). Incentives for rent-seeking, inefficiency, and corruption drove the actual working of federations, especially in contexts where sub-national units could spend resources partly derived from other parts of the country without significant oversight.

In contrast, our paper shares the message of earlier models, at least if one's objective is minimizing the polarization of national parties. We suggest that decentralized solutions are better at handling polarized political geographies, since Geographic Balancing between the national government and the regions exists *regardless of the levels of regional ideological heterogeneity.* Classical theories suggest decentralized solutions are efficient in managing polities with heterogeneous and concentrated preferences. Our results confirm this idea, and point to potential decentralized strategies to handle divisive issues with a geographic gradient.

By contrast, when the center overpowers the regions, the institutional accommodation of a polarized political geography becomes more challenging. Centralization, when combined with sufficiently high penalties on party disunity, may lead to the high-jacking of national platforms by regional extremists. Under these conditions, moderation is sacrificed at the altar of party unity. The result is the elimination of internal party heterogeneity, the translation of regional extremism to the national arena, and the consolidation of a political dynamic which empowers extremists. Indeed, not only may centralized systems under-perform when faced with a polarized political geography; they may also under-perform when faced with a *highly moderate* electorate. In these circumstances, if state institutions are sufficiently centralized, we cannot rule an Asymmetric form of polarization that pits regional extremists against the center.

Our core model of party competition and voter choice should be applicable to theoretical questions and contexts beyond the scope of this paper. Most immediately, we look forward to extending our theoretical framework to environments in which centralization is modeled not as a win-bonus for regional politicians, but rather as an institutional process in which platforms/policies are a weighted function of inputs from distinct territorial elections (Dixit and Londregan 1998; Cremer and Palfrey 1999). While the full analysis lies ahead, we have reason to suspect that an insight similar to that captured in Theorem 1 will emerge, allowing us to develop additional results regarding the relationship between political institutions, political geography, and the dynamic of regional capture.

A second natural extension would be to political-economy models of public goods, distributive politics, and rent-seeking (Dixit and Londregan 1996; Persson and Tabellini 2001). In these models candidate choice would involve a vector of fiscal policy decisions, and voter choice would be driven by their consumption of public and club goods, as well as their disutility for rent-seeking. Regional candidates would value office at both territorial levels, and voters would penalize party disunity in economic policy proposals, due for example to a reduced sense that divided parties can genuinely deliver on economic policy promises. Such a model would allow us to evaluate how political geography and centralization affect the equilibrium levels of public goods, pork, and political graft.

Finally, we believe our modeling framework could be applied to multi-branch elections rather than multi-level elections. The balance of power between the legislature and the presidency would determine the value that candidates of one branch place on having their party control the other; and voters would punish parties in the executive contest when their legislative candidate diverges from the party's presidential candidate (and maybe vice versa). This specification would allow us to examine how distinct design, agenda, and veto rules contribute to the mitigation or exacerbation of geographic polarization in the electorate. In short, beyond the immediate findings, we believe the paper opens a path to numerous lines of additional theoretical research, which we hope to tackle in subsequent contributions.

References

Alesina, A. and H. Rosenthal (1996). A theory of divided government. Econometrica: journal of the Econometric Society, 1311–1341.

Aragones, E. and T. R. Palfrey (2002). Mixed equilibrium in a downsian model with a favored candidate. Journal of Economic Theory 103(1), 131–161.

Ashworth, S. and E. B. De Mesquita (2009). Elections with platform and valence competition. Games and Economic Behavior 67(1), 191–216.

Austen-Smith, D. (1981). Party policy and campaign costs in a multi-constituency model of electoral competition. Public Choice 37(3), 389–402.

Austen-Smith, D. (1984). Two party competition with many constituencies. Mathematical Social Sciences 7, 177–198.

Balart, P., A. Casas, and O. Troumpounis (2022). Technological change, campaign spending and polarization. Journal of Public Economics 211, 104666.

Bednar, J. et al. (2009). The robust federation. Cambridge Books.

Beramendi, P. (2012). The political geography of inequality: regions and redistribution. Cambridge University Press.

Beramendi, P. and M. Rogers (2022). Geography, capacity, and inequality: Spatial inequality. Cambridge University Press.

Bernhardt, D., P. Buisseret, and S. Hidir (2020). The race to the base. American Economic Review 110(3), 922–942.

Callander, S. (2005). Electoral competition in heterogeneous districts. Journal of Political Economy 113(5), 1116–1145.

Carrillo, J. D. and M. Castanheira (2008). Information and strategic political polarisation. The Economic Journal 118(530), 845–874.

Ceron, A. and E. Volpi (2022). How do parties react to defections? electoral strategies after a valence loss. European Journal of Political Research 61(4), 1042–1059.

Clark, M. (2009). Valence and electoral outcomes in western europe, 1976-1998. Electoral Studies 29, 111–122.

Clark, M. (2014). Does public opinion respond to shifts in party valence? a cross-national analysis of western europe, 1976–2002. West European Politics 37(1), 91–112.

Cremaschi, S., P. Rettl, M. Cappelluti, and C. E. De Vries (2024). Geographies of discontent: Public service deprivation and the rise of the far right in italy. American Journal of Political Science.

Crémer, J. and T. R. Palfrey (1999). Political confederation. American Political Science Review 93(1), 69–83.

Crutzen, B. S., M. Castanheira, and N. Sahuguet (2010). Party organization and electoral competition. The Journal of Law, Economics, & Organization 26(2), 212–242.

Dixit, A. and J. Londregan (1996). The determinants of success of special interests in redistributive politics. Journal of Politics 58(4), 1132–1155.

Dixit, A. and J. Londregan (1998). Fiscal federalism and redistributive politics. Journal of Public Economics 68(2), 153–180.

Greene, Z. D. and M. Haber (2015). The consequences of appearing divided: An analysis of party evaluations and vote choice. Electoral Studies 37, 15–27.

Groseclose, T. (2001). A model of candidate location when one candidate has a valence advantage. American Journal of Political Science, 862–886.

Iaryczower, M. and A. Mattozzi (2012). The pro-competitive effect of campaign limits in non-majoritarian elections. Economic Theory 49(3), 591–619.

Iaryczower, M. and A. Mattozzi (2013). On the nature of competition in alternative electoral systems. The Journal of Politics 75(3), 743–756.

Iversen, T. and D. Soskice (2019). Democracy and prosperity: Reinventing capitalism through a turbulent century.

Izzo, F. (2023). Ideology for the future. American Political Science Review 117(3), 1089–1104.

Kittsteiner, T. and E. Eyster (2007). Party platforms in electoral competition with heterogeneous constituencies. Theoretical Economics 2(1), 41–70.

Krasa, S. and M. Polborn (2018). Political competition in legislative elections. American Political Science Review 112(4), 809–825.

Lehrer, R., P. Stöckle, and S. Juhl (2022). Assessing the relative influence of party unity on vote choice: evidence from a conjoint experiment. Political Science Research and Methods, 1–9.

Oates, W. E. (1972). Fiscal Federalism. New York: Harcourt Brace Jovanovich.

Persson, T., G. Roland, and G. Tabellini (1997). Separation of powers and political accountability. The Quarterly Journal of Economics 112(4), 1163–1202.

Persson, T. and G. Tabellini (1994). Does centralization increase the size of government? European Economic Review 38(3-4), 765–773.

Persson, T. and G. Tabellini (1996). Federal fiscal constitutions: Risk sharing and moral hazard. Econometrica: Journal of the Econometric Society, 623–646.

Persson, T. and G. Tabellini (2001). Political Economics: Explaining Economic Policy. MIT Press.

Polborn, M. and J. M. Snyder Jr (2017). Party polarization in legislatures with office-motivated candidates. Quarterly Journal of Economics 33(3), 1509–1550.

Rodden, J. (2006). Hamilton's paradox: the promise and peril of fiscal federalism, Volume 2. Cambridge University Press Cambridge.

Rodden, J. (2010). The geographic distribution of political preferences. Annual Review of Political Science 13(1), 321–340.

Rodden, J. A. (2019). Why cities lose: The deep roots of the urban-rural political divide. Hachette UK.

Rodríguez-Pose, A. (2018). The revenge of the places that don't matter (and what to do about it). Cambridge journal of regions, economy and society 11(1), 189–209.

Serra, G. (2010). Polarization of what? a model of elections with endogenous valence. The Journal of Politics 72(2), 426–437.

Tiebout, C. M. (1956). A pure theory of local expenditures. Journal of political economy 64(5), 416–424.

Weingast, B. R. (2014). Second generation fiscal federalism: Political aspects of decentralization and economic development. World Development 53, 14–25.

Weingast, B. R. et al. (2005). Self-enforcing federalism. Journal of Law, Economics & Organization 21(1).

Wibbels, E. (2005). Federalism and the market: Intergovernmental conflict and economic reform in the developing world. Cambridge University Press.

Zakharov, A. V. (2009). A model of candidate location with endogenous valence. Public Choice 138(3), 347–366.

Zhou, C. (2025). Interactions among simultaneous elections. Political Science Research and Methods, 1–13.

Theoretical Appendix

A Baseline Results

Proof of Lemma 2

If $x_B^N > x_A^N$, then the utility differential $u_i^B(\mathbf{x}) - u_i^A(\mathbf{x}) = (x_i^R - x_A^N)^2 - (x_i^R - x_B^N)^2 + k\Big(|x_A^R - x_A^N| - |x_B^R - x_B^N|\Big) - \sigma \equiv D^R$. In turn, it is straightforward to show that:

$$\frac{\Delta D^R}{\Delta x_i^R} = 2(x_B^N - x_A^N) > 0. \tag{A1}$$

Since the utility differential $u_i^B(\mathbf{x}) - u_i^A(\mathbf{x})$ is increasing in x_i^R, if a voter with ideal point x_i^R prefers the 'higher' platform x_B^N, then so does any voter j with $x_j^R > x_i^R$; and if a voter with ideal point x_i^R prefers the 'lower' platform x_A^N, then so does any voter j with $x_j^R < x_i^R$. **QED**

Deriving the Swing Voter

x_s^R is defined implicitly by the equation $-k\big(|x_A^R - x_A^N|\big) - (x_s^R - x_A^N)^2 + \sigma = k\big(|x_B^R - x_B^N|\big) - (x_s^R - x_B^N)^2$. This can be rearranged to:

$$x_s^R = \left(\frac{k\big(|x_B^R - x_B^N| - |x_A^R - x_A^N|\big) + (x_B^N)^2 - (x_A^N)^2 + \sigma}{2(x_B^N - x_A^N)}\right). \tag{A2}$$

In turn, using the fact that $(x_B^N)^2 - (x_A^N)^2 = (x_B^N - x_A^N)(x_B^N + x_A^N)$, we can rearrange once again to derive equation (5). **QED**

A.1 Proof of Theorem 1

Lemma A1 *As long as $k < 1$ candidates A^N and B^N can make $\pi_P^N(\mathbf{x}) > 0$ by choosing the national median voter's ideal point $x_m^N = 1$ as a platform. When $k = 1$ the same is true in all cases except the strategy vectors where : i) $x_B^1 = x_B^2 = x_B^N = 1$ and $\Delta_A = 2$; or ii.) $x_A^1 = x_A^2 = x_A^N = 1$ and $\Delta_B = 2$.*

Proof of Lemma A1

Consider candidate A^N. To prove that by choosing $x_A^N = 1$ she can make $\pi_A^N(\mathbf{x}) > 0$ we construct the *hardest possible case* for this to occur, i.e. the case in which B^N and the four regional candidates adopt platforms which give B^N the greatest advantage in the national contest, making it the least likely that $\pi_A^N(\mathbf{x}) > 0$. From (10) in the text we know that: a.) all else equal $\pi_B^N(\mathbf{x})$ increases as x_B^N moves towards the national median at $x_m^N = 1$; and b.) all else equal $\pi_B^N(\mathbf{x})$ increases as Δ_B decreases. In turn, the set of platforms in party B which makes it most difficult to satisfy the condition $\pi_A^N(\mathbf{x}) > 0$ is $x_B^1 = x_B^2 = x_B^N = 1$.

Similarly, we know that all else equal $\pi_A^N(\mathbf{x})$ decreases as Δ_A increases. Recall that by construction $x_A^N \leq 1$. In turn, for any $x_A^N < 1$ the maximum Δ_A is achieved when $x_A^1 = x_A^2 = 2$, which makes $\Delta_A = 2(2 - x_A^N)$. If $x_A^N = 1$ then there are four distinct sets of regional platforms which generate the maximum distance: $x_A^1 = x_A^2 = 2$, $x_A^1 = x_A^2 = 0$, $x_A^1 = 0$ while $x_A^2 = 2$, and $x_A^1 = 2$ while $x_A^2 = 0$. In all four $\Delta_A = 2$.

As such, for any $x_A^N \leq 1$ the set of platforms $x_B^N = x_B^1 = x_B^2 = 1$ and $x_A^1 = x_A^2 = 2$ makes it (weakly) most difficult to satisfy the condition $\pi_A^N(\mathbf{x}) > 0$, since this maximizes B's party unity and proximity to the national median, and also (weakly) maximizes party A's *dis*unity and distance from the national median. Given that all other candidates choose these platforms, the platform which gives A^N the best chance of making $\pi_A^N(\mathbf{x}) > 0$ is $x_A^N = 1$, since this platform maximizes her proximity to the spatial median, and since her platform choice has no affect on party unity $\Delta_A = 2$.[A1] In turn, by fixing all other platforms at their 'hardest possible case' values of $x_B^N = x_B^1 = x_B^2 = 1$ and $x_A^1 = x_A^2 = 2$, and setting $x_A^N = 1$, we can use (10) from the text to write:

$$\pi_A^N(\mathbf{x}) = \frac{1}{2} - \frac{k}{2}, \tag{A3}$$

where the second and third terms from (10) drop out (since $x_A^N = x_B^N = 1$) and where the last term results from the fact that $\Delta_B = 0$ and $\Delta_A = 2$. In turn, for this hardest case scenario

$$\frac{1}{2} - \frac{k}{2} > 0 \Rightarrow k < 1. \tag{A4}$$

Thus for any $k < 1$, even in the worst case scenario candidate A^N can choose $x_A^N = 1$ and ensure herself a positive probability of winning the national contest. If all other candidates choose the hardest possible case platforms and $k = 1$, then when A^N chooses $x_A^N = 1$ (A3) is exactly 0 and A^N cannot secure a positive probability of winning the national contest. However, from (10) in the text we know that, at any strategy vector other than the 'hardest possible case' strategy vectors, candidate A^N can guarantee that $\pi_A^N(\mathbf{x} > 0)$ by choosing $x_A^N = 1$ even if $k = 1$, since at any other strategy vector either $x_B^N \neq 1$, $\Delta_B > 0$, and/or $\Delta_A < 2$. **QED**

Henceforth, for $P \in \{A, B\}$ we will use the notation x_P^N and x_P^R to denote the platforms adopted by candidates P^N and P^R at a stipulated strategy vector $\mathbf{x}$; while we let $\hat{x}_P^N$ and $\hat{x}_P^R$ represent some deviation from that stipulated strategy vector. Similarly, $\pi_P^N(\mathbf{x})$ and $\pi_P^R(\mathbf{x})$ will represent the relevant probabilities of winning at a stipulated strategy vector $\mathbf{x}$, while $\hat{\pi}_P^N(\cdot)$ and $\hat{\pi}_P^R(\cdot)$ represent the new probabilities given some platform deviation by the relevant candidate. Finally, we will use the term $\epsilon \to 0$ to represent an infinitesimally small number.

Lemma A2 *In any equilibrium such that $x_A^R \neq x_B^R$, it must be the case that $x_A^R = x_A^N$ and $x_B^R = x_B^N$.*

Proof of Lemma A2

Note first that there will never be an equilibrium in which x_A^R and x_B^R are symmetric around the regional median x_m^R. Consider the case in which $x_A^1 < x_m^1 < x_B^1$ and

[A1]To see the latter, note that any for any $x_A^N \in [x_A^1, x_A^2]$, $\Delta_A = (x_A^2 - x_A^N) + (x_A^N - x_A^!) = (x_A^2 - x_A^1) = 2$, i.e A^N can choose any platform in that range without affecting party unity, and is thus free to choose the national spatial median.

$(x_m^1 - x_A^1) = (x_B^1 - x_m^1)$. In this case, both candidates win the regional contest with $\pi_P^1(\mathbf{x}) = \frac{1}{2}$. In turn, A^1 (B^1) could make an infinitesimal deviation to the platform $\hat{x}_A^1 = x_A^1 + \epsilon$ ($\hat{x}_B^1 = x_B^1 - \epsilon$) and win the local contest with certainty, at only infinitesimal cost to her probability of winning the national contest. We know from (2) in the text that such a deviation is profitable. The same applies in region 2. Therefore, among cases such that $x_A^R \neq x_B^R$ we need only examine cases in which either A^R or B^R wins the regional election outright.

Part 1: Looking again at region 1, consider a strategy vector $\mathbf{x}$ at which platforms satisfy the following conditions: $x_A^1 \neq x_B^1$, $x_A^1 \neq x_A^N$ and $x_B^1 \neq x_B^N$. Without loss of generality, let $\left|x_A^1 - x_m^1\right| < \left|x_B^1 - x_m^1\right|$ such that A^1 wins the regional contest in region 1. We can exhaustively assign all strategy vectors $\mathbf{x}$ which meet these criteria to one of three possible cases, according to the two parties' probabilities of winning the national contest $\pi_P^N(\mathbf{x})$.

Case 1a: $0 < \pi_A^N(\mathbf{x}), \pi_B^N(\mathbf{x}) < 1$. In this case, candidate B^1 has the incentive to deviate to some platform $\hat{x}_B^1$ which is closer to x_B^N, and increase their probability of winning the national election $\hat{\pi}_B^N(\cdot)$, while leaving them no worse off in the regional election (they were losing in any case).

Case 1b: $\pi_B^N(\mathbf{x}) = 0$. In this case, candidate B^1 receives a payoff of 0 at $\mathbf{x}$, and has the incentive to deviate to a position $\hat{x}_B^1$ just barely closer to x_m^1 than x_A^1 so as to win the regional election outright; or to the position $\hat{x}_B^1 = x_m^1$ in the event that $x_A^1 = x_m^1$ so as to win the regional election with probability $\hat{\pi}_B^1(\cdot) = \frac{1}{2}$. This deviation allows B^1 to win (or tie) region 1's election while leaving them no worse off in the national election (they were losing in any case).

Case 1c: $\pi_A^N(\mathbf{x}) = 0$. In this case, we cannot guarantee that B^1 has a deviation: deviating to $\hat{x}_B^1$ so as to win or tie the regional election could increase Δ_B and in turn reduce their probability of winning the national contest to $\hat{\pi}_B^N(\cdot) < 1$. However, by construction A^1 wins the regional election, meaning that $\left|x_A^N - x_m^1\right| < \left|x_B^N - x_m^1\right|$. In turn, by Lemma A1 we know that candidate A^N has a profitable deviation to the position $\hat{x}_A^N = 1$.

A qualitatively identical proof applies in the event that $\left|x_B^1 - x_m^1\right| < \left|x_A^1 - x_m^1\right|$ such that B^1 wins the regional contest in region 1. Thus, there will always be at least one candidate that deviates from a strategy vector such that $x_A^1 \neq x_B^1$, $x_A^1 \neq x_A^N$, $x_B^1 \neq x_B^N$.

Part 2: Now consider a strategy vector $\mathbf{x}$ at which platforms satisfy the following conditions: $x_A^1 \neq x_B^1$, $x_A^1 \neq x_A^N$ and $x_B^1 = x_B^N$, and once again without loss of generality let $\left|x_A^1 - x_m^1\right| < \left|x_B^1 - x_m^1\right|$ such that candidate A^1 wins the regional contest in region 1. This situation differs from **Part 1** only in that $x_B^1 = x_B^N$ rather than $x_B^1 \neq x_B^N$. We can once again exhaustively assign all strategy vectors $\mathbf{x}$ which meet these criteria to one of three possible cases:

Case 2a: $0 < \pi_A^N(\mathbf{x}), \pi_B^N(\mathbf{x}) < 1$. In this case A^1 will always have at least a small deviation to a platform $\hat{x}_A^1$ slightly closer to x_A^N, that increases A^N's probability of winning the national election $\hat{\pi}_A^N(\cdot)$ without costing them the regional contest, and this deviation will be profitable.

Case 2b: $\pi_B^N(\mathbf{x}) = 0$. This case is identical to Case (1b) above.

Case 2c: $\pi_A^N(\mathbf{x}) = 0$. This case is identical to Case (1c) above.

A qualitatively identical proof applies in the event that $\left|x_B^1 - x_m^1\right| < \left|x_A^1 - x_m^1\right|$ such that B^1 wins the regional contest in region 1. Thus, there will always be at least one candidate that deviates from a strategy vector such that $x_A^1 \neq x_B^1$, $x_A^1 \neq x_A^N$, $x_B^1 = x_B^N$.

Part 3: Finally, consider a strategy vector $\mathbf{x}$ at which platforms satisfy the following conditions: $x_A^1 \neq x_B^1$, $x_A^1 = x_A^N$ and $x_B^1 \neq x_B^N$, and once again without loss of generality let $\left|x_A^1 - x_m^1\right| < \left|x_B^1 - x_m^1\right|$ such that candidate A^1 wins the regional contest in region 1. This situation only differs from **Part 1** in that $x_A^1 = x_A^N$ rather than $x_A^1 \neq x_A^N$. The proof that a profitable deviation exists in all cases for at least one candidate is identical to that for **Part 1**, and omitted for redundancy.

Finally, taking cases (1), (2), and (3) together establishes that anytime $x_A^1 \neq x_B^1$ it must be the case that $x_A^1 = x_A^N$ and $x_B^1 = x_B^N$, as in any other possible situation at least one candidate will have a profitable deviation. Via symmetry the proof applies equally to region 2, thus establishing Lemma A2. **QED**

Corollary A1 *If $x_A^1 \neq x_B^1$ and $x_A^2 \neq x_B^2$, then in any equilibrium it must be the case that parties are perfectly unified: $x_P^1 = x_P^2 = x_P^N$ for $P \in \{A, B\}$.*

Corollary A1 follows directly from Lemma A2: since there cannot be a NE in which regional candidates diverge and announce distinct policies from their national party, it must be the case that, if regional candidates in both regions diverge they both choose their national platform, creating perfect Party Unity $x_P^1 = x_P^2 = x_P^N$.

We finalize the proof of Theorem 1 with the following Lemma, which establishes that there is only one possible equilibrium with disunified parties, which we label in the text the Geographic Balancing strategy vector $\mathbf{x^G}$.

Lemma A3 *The only strategy vector at which $x_A^R = x_B^R \ \exists\, R \in \{1, 2\}$ is $x_A^1 = x_B^1 = x_m^1$, $x_A^2 = x_B^2 = x_m^2$, and $x_A^N = x_B^N = 1$.*

Proof of Lemma A3

Note first that anytime $x_A^R = x_B^R$, in any equilibrium it must be the case that $x_A^R = x_B^R = x_m^R$, since otherwise either A^R or B^R could make an infinitesimal deviation towards the regional median, having essentially no effect on their probability of winning the national contest $\hat{\pi}_P^N(\cdot)$, and allowing them to win the regional election with $\hat{\pi}_P^R(\cdot) = 1$ rather than with probability $\pi_P^R(\mathbf{x}) = \frac{1}{2}$.

Note second that there is never an equilibrium in which $x_A^R = x_B^R = x_m^R$ in one region but $x_A^{\sim R} \neq x_B^{\sim R}$ in the other region; i.e. where regional candidates converge in one region but diverge in the other. To see this, without loss of generality let $x_A^1 = x_B^1 = x_m^1$ and assume $x_A^2 \neq x_B^2$, which via Lemma A2 implies that $x_A^2 = x_A^N$ and $x_B^2 = x_B^N$. In this case, for any $x_B^N = x_B^2 > 1$ we know from (10) in text that

B^N will have an optimal deviation to $\hat{x}_B^N = 1$, which increases her proximity to the spatial median without causing additional organizational disunity.[A2]

For the case in which $x_B^N = x_B^2 = 1$ candidate B^N does not have a profitable deviation. Via Lemma A2 an equilibrium requires $x_A^2 = x_A^N \leq 1$, and by construction $x_A^2 \neq x_B^2$, which means that if $\mathbf{x}$ is an equilibrium candidate A^2 loses her regional election. Define $\hat{\pi}_A^N(\cdot)$ as the probability that A^N wins the national contest if candidate A^2 deviates to the platform $\hat{x}_A^2 = 1 + \epsilon$, just barely closer to the regional median x_m^2 than is $x_B^2 = 1$. This deviation is optimal if $\pi_A^N(\mathbf{x})\beta < 1 + \hat{\pi}_A^N(\cdot)\beta$, which can be re-written using (10) from the text as follows:

$$\beta < \frac{4}{k(1 - x_A^N)}. \tag{A5}$$

Now consider candidate B^1 who ties her regional election at $\mathbf{x}$. Define $\hat{\pi}_B^N(\cdot)$ as the probability that B wins the national contest if candidate B^1 deviates to the platform $\hat{x}_B^2 = 1$ which minimizes $\Delta_B = 0$. This deviation is optimal if $\pi_B^N(\mathbf{x})\beta + \frac{1}{2} < \hat{\pi}_B^N(\cdot)\beta$, which can be rewritten using (10) from the text as follows:

$$\beta > \frac{2}{k(1 - x_m^2)}. \tag{A6}$$

Finally, note that $\frac{2}{k(1-x_m^2)} < \frac{4}{k(1-x_A^2)}$, which means that either (A6) or (A7) must always hold, i.e. that for the case in which $x_B^N = x_B^2 = 1$ either candidate A^2 or candidate B^1 will always have a profitable deviation. This establishes that there is never an equilibrium in which $x_A^1 = x_B^1 = x_m^1$ and $x_A^2 \neq x_B^2$. An identical proof applies to the case in which $x_A^2 = x_B^2 = x_m^2$ and $x_A^1 \neq x_B^1$.

Taken together we have shown that anytime $x_A^R = x_B^R$ it must be the case that $x_A^R = x_B^R = x_m^R$, and that it can never be the case that $x_A^R = x_B^R = x_m^R$ in one region while $x_A^{\tilde{R}} \neq x_B^{\tilde{R}}$ in the other. Thus, the only remaining strategy vector at which candidates A^R and B^R choose identical platforms is that at which $x_A^1 = x_B^1 = x_m^1$ and $x_A^2 = x_B^2 = x_m^2$. In this case, both national candidates maximize their utility at $x_A^N = x_B^N = 1$, which (weakly) minimizes their disunity $\Delta_P = x_m^2 - x_m^1 \equiv d$,[A3] while maximizing their proximity to the spatial median. This completes the proof of Lemma A3. **QED**

Together Lemma A2, Corollary A1, and Lemma A3 establish Theorem 1. **QED**

A.2 Geographic Balancing

Proof of Proposition 1

At the strategy vector $\mathbf{x^G}$ all four regional candidates win their regional election with $\pi_P^R(\mathbf{x^G}) = \frac{1}{2}$ (Lemma 1); and via (10) in the text we know that both parties

[A2]To see the latter, note that any for any $x_B^N \in [1, x_B^2]$, $\Delta_B = (x_B^2 - x_B^N) + (x_B^N - 1) = (x_B^2 - 1)$, i.e B^N can choose any platform in that range without affecting party unity, and is thus free to choose the national spatial median.

[A3]To see this note that for any $x_A^N \in [x_m^1, 1]$ party A's disunity is $\Delta_A = (x_m^2 - x_A^N) + (x_A^N - x_m^1) = x_m^2 - x_m^2$, i.e. disunity for is equal at any platform in between the two regional medians, allowing A^N to choose the spatial median $x_m^N = 1$. The same is true for B^N.

win the national election with $\pi_P^N(\mathbf{x^G}) = \frac{1}{2}$: they choose identical platforms ($x_B^N = x_A^N = 1$) and have identical disunity ($\Delta_A = \Delta_B = x_m^2 - x_m^1 \equiv d$). We know that neither A^N nor B^N has an optimal deviation from this strategy vector: when all four regional candidates choose their regional median, the platform $x_P^N = 1$ *both* maximizes spatial proximity to the national median *and* (weakly) minimizes organizational disunity (ftn A3).

At $\mathbf{x^G}$ all four regional candidates receive a payoff of $U_P^R(\mathbf{x^G}) = \left(\frac{1+\beta}{2}\right)$. Without loss of generality consider candidate A^1. Trivially, any deviation to a position $\hat{x}_A^1 < x_m^1$ or $\hat{x}_A^1 \geq x_m^2$ is not profitable: it would cost the candidate the regional election and decrease her probability of winning the national election. Any deviation to $\hat{x}_A^1 \in (x_m^1, x_m^2]$ would cost the deviating candidate the regional election, but would also decrease Δ_A and thus increase $\hat{\pi}_A^N(\cdot)$. Of all such deviations, that which would create the greatest increase in $\hat{\pi}_A^N(\cdot)$ is $\hat{x}_A^1 = 1$, i.e. that at which A^1 eliminates all disunity between herself and A^N. From (10) in the text, we know that a unilateral deviation from $x_A^1 = x_m^1$ to $\hat{x}_A^1 = 1$ yields A^1 the payoff:

$$\hat{U}_A^1 = \beta \cdot \left[\frac{1}{2} + \frac{k}{4}(1 - x_m^1)\right]. \tag{A7}$$

In turn, we derive the condition for Proposition 1 as follows:

$$U_A^1(\mathbf{x^G}) > \hat{U}_A^1(\cdot) \Rightarrow \left(\frac{1+\beta}{2}\right) > \beta \cdot \left[\frac{1}{2} + \frac{k}{4}(1 - x_m^1)\right] \Rightarrow \beta < \frac{4}{kd}. \quad \textbf{QED} \tag{A8}$$

B Party Unity Nash Equilibria

Remark A1 *At any Party Unity strategy vector* $\mathbf{x}$ *such that* $0 < x_A^N \leq 1 \leq x_B^N < 2$ *it must be the case that* $\pi_A^N(\mathbf{x}), \pi_B^N(\mathbf{x}) > 0$.

To see this, note that from among the possible Party Unity strategy vectors, those which give one party the greatest advantage over the other in the national contest are: i.) $x_A^N = x_A^1 = x_A^2 = 0$ and $x_B^N = x_B^1 = x_B^2 = 1$; ii.) $x_A^N = x_A^1 = x_A^2 = 1$ and $x_B^N = x_B^1 = x_B^2 = 2$. Without loss of generality consider (i), at which A^N's probability of winning the national election is:

$$\pi_A^N = \frac{1}{2} - (1-0) + \frac{(1-0)(1+0)}{2} = 0, \tag{A9}$$

meaning that in her most disadvantaged situation A^N has exactly $\pi_A^N(\mathbf{x}) = 0$; and that at any minutely different Party Unity strategy vector where $x_A^N = 0 + \epsilon$ or $x_B^N = 1 + \epsilon$, this probability will increase to $\pi_A^N(\mathbf{x}) > 0$. An identical argument applies for B^N, establishing the Remark. This obviates the need to consider deviations from situations in which a national candidate has no chance of winning.

Proof of Lemma 3

From (3) and (10) we know that Candidate A^N's utility at any Party Unity strategy vector $\mathbf{x}$ is:

$$U_A^N(\mathbf{x}) = \pi_A^N(\mathbf{x}) = \frac{1}{2} - (x_B^N - x_A^N) + \frac{(x_B^N - x_A^N)(x_B^N + x_A^N)}{2}. \tag{A10}$$

Trivially any deviation for A^N to a platform $\hat{x}_A^N < x_A^N$ is not profitable, as it creates both greater spatial distance from the national median and greater organizational disunity. The disunity created by deviating to some platform $\hat{x}_A^N \in (x_A^N, 1]$ will be $\Delta_A = 2(\hat{x}_A^N - x_A^N)$, and the deviation utility is written:

$$\hat{U}_A^N(\cdot) = \hat{\pi}_A^N(\cdot) = \frac{1}{2} - (x_B^N - \hat{x}_A^N) + \frac{(x_B^N - \hat{x}_A^N)(x_B^N + \hat{x}_A^N)}{2} - \frac{k(\hat{x}_A^N - x_A^N)}{2}. \quad \text{(A11)}$$

This deviation will be optimal if (A11)>(A10), which after rearrangement can be written as:

$$k < 2 - \frac{(x_B^N - x_A^N)(x_B^N + x_A^N) - (x_B^N - \hat{x}_A^N)(x_B^N + \hat{x}_A^N)}{(\hat{x}_A^N - x_A^N)}. \quad \text{(A12)}$$

Note now that the numerator of the second term in (A12) can be rewritten as:

$$[(x_B^N)^2 - (x_A^N)^2] - [(x_B^N)^2 - (\hat{x}_A^N)^2] = (\hat{x}_A^N)^2 - (x_A^N)^2 = (\hat{x}_A^N - x_A^N)(\hat{x}_A^N + x_A^N), \quad \text{(A13)}$$

which means that (A12) as a whole can be rewritten as:

$$k < 2 - (\hat{x}_A^N + x_A^N). \quad \text{(A14)}$$

If this inequality is satisfied, A^N's deviation to $\hat{x}_A^N$ is profitable. This means that the closer is $\hat{x}_A^N$ to x_A^N the more likely the deviation to $\hat{x}_A^N$ is profitable (the more likely (A14) is satisfied). To understand this, note that k captures the marginal cost of deviating to $\hat{x}_A^N$, which is linear due to the absolute value loss function for party disunity. In contrast, the marginal benefit for deviating to $\hat{x}_A^N$ (LHS of (A14)) is decreasing as $\hat{x}_A^N \to 1$, due to the quadratic shape of voters' spatial loss function. In words, while any deviation towards $x_m^N = 1$ increases the national median voter's spatial utility for A^N, these increases come at a decreasing rate as $\hat{x}_A^N \to 1$.

As such, the deviation with the highest marginal benefit in terms of spatial competitiveness is that to $\hat{x}_A^N = x_A^N + \epsilon$, i.e. this is the deviation that is most likely to be profitable, allowing us to rewrite (A14) as:

$$k < 2 - 2x_A^N \;\; \Rightarrow \;\; x_A^N < 1 - \frac{k}{2}. \quad \text{(A15)}$$

If this condition is met, then A^N has a profitable deviation from $\mathbf{x}$; if not, then no such deviation exists. **QED**

Note that, since by assumption $k \leq 1$, this fixes the range of possible Party Unity Nash Equilibria to those in which $x_A^N \geq \frac{1}{2}$. An identical argument demonstrates that the condition for B^N's deviation is $x_B^N > 1+\frac{k}{2}$, which fixes the range of possible Party Unity Nash Equilibria to those in which $x_B^N \leq \frac{3}{2}$.

Remark A2 *For any Party Unity strategy vector* $\mathbf{x}$ *it is never possible that* B^1 *wins in region* 1 *while* A^2 *wins in region* 2.

Recall that by construction $x_A^N \leq 1 \leq x_B^N$. In turn, at any Party Unity strategy vector, we know that for B^1 to win in region 1 it must be the case that: i) $x_A^N < x_m^1$ and ii) $x_B^N < 2x_m^1 - x_A^N$. Conversely, if $x_A^N \geq x_m^1$ or $x_B^N \geq 2x_m^1 - x_A^N$, then A^1 wins (or at least ties) in region 1's election. Similarly, at any Party Unity strategy vector, for A^2 to win in region 2 it must be the case that: i) $x_B^N > x_m^2$ and ii) $x_A^N > 2x_m^2 - x_B^N$. Note by the two conditions we have $x_B^N < 2x_m^1 - x_A^N$ and $x_B^N > 2x_m^2 - x_A^N$, which leads to a contradiction since $x_m^1 < x_m^2$ by construction. This establishes the Remark.

Recall from the proof of Lemma A2 that there is a never an equilibrium in which regional candidates position symmetrically around the regional median. Combined with Remark 1 from the text, this implies that there is never an equilibrium in which two regional candidates each win the regional contest with probability $\pi_A^R = \pi_B^R = \frac{1}{2}$. In turn, this fact combined with Remark A2 allows us divide the set of Party Unity strategy vectors which can possibly constitute equilibria into one of two basic categories: a.) those at which A^1 wins in region 1 and B^2 wins in region 2, and b.) those in which one of the two parties wins both regional elections. We label the first 'Type 1' strategy vectors and the second 'Type 2' strategy vectors.

B.1 Type 1 Party Unity Nash Equilibria

Proof of Proposition 2

Candidates A^1 and B^2 do not have a profitable deviation from $\mathbf{x^R}$, as they win their respective regional elections and thus do not have any incentive to create disunity in their organization. Furthermore, recall that by assumption $x_m^1 = 2 - x_m^2$ and that we define $d = x_m^2 - x_m^1$. Hence, we know that $d \leq k \Rightarrow x_m^2 < 1 + \frac{k}{2} \equiv \bar{x}_k$. In turn, if $d \leq k$ then from Lemma 3 we know that B^N does not deviate from $\mathbf{x^R}$. An identical argument applies to A^N.

Turn now to candidates A^2 and B^1, who lose their regional elections at $\mathbf{x^R}$. Note that $\pi_A^N(\mathbf{x^R}) = \frac{1}{2}$, since the two parties are equidistant from the national median and has any disunity. In turn, candidate A^2's utility at $\mathbf{x^R}$ is simply $U_A^2(\mathbf{x}) = \frac{\beta}{2}$. Their only possible profitable deviation is to $\hat{x}_A^2 = x_m^2$, which allows them to win the regional contest with probability $\hat{\pi}_A^2(\cdot) = \frac{1}{2}$ (since at $\mathbf{x^R}$, $x_B^2 = x_m^2$); any other deviation reduces A^N's probability of winning the national election without changing the regional outcome, and is not profitable for A^2. From (2) and (10) we can write their utility for deviating to $\hat{x}_A^2 = x_m^2$ as:

$$\hat{U}_A^2(\cdot) = \frac{1}{2} + \beta\left[\frac{1}{2} - \frac{kd}{4}\right], \tag{A16}$$

which in turn implies that deviating is profitable if:

$$\frac{1}{2} + \beta\left[\frac{1}{2} - \frac{kd}{4}\right] > \frac{\beta}{2} \Rightarrow \beta < \frac{2}{dk}. \tag{A17}$$

An identical analysis shows that B^1's deviation to x_m^1 is also profitable if and only if $\beta < \frac{2}{dk}$. When combined with the analysis of candidates A^N and B^N above, this establishes Proposition 2. **QED**

Proof of Proposition 3

Via Lemma 3 neither A^N nor B^N has a profitable deviation from $\mathbf{x^S}$, since $x_A^N = \underline{x}_k$ and $x_A^N = \overline{x}_k$. Furthermore, neither A^1 nor B^2 has a profitable deviation from $\mathbf{x^S}$, as they win their respective regional elections. We divide the proof for candidates A^2 and B^1 into two cases: a) $d > k$ and b) $d \leq k$.

Part 1: $d > k$. Note that $\pi_A^N(\mathbf{x^S}) = \frac{1}{2}$, since path parties are equidistant from the national median and have no disunity. In turn A^2's utility at $\mathbf{x^S}$ is simply $\frac{\beta}{2}$. When $d > k$ their optimal deviation is to the platform $\hat{x}_A^2 = \bar{x}_k + \epsilon$, which allows them to just barely win their regional contest against B^2 while minimizing the organizational disunity they create. Any other deviation does not allow them to win region 2's election outright ($\hat{x}_A^2 \in (\underline{x}_k, x_m^2]$) or does so at unnecessary cost ($\hat{x}_A^2 \in (x_m^2, 2]$). From (2) and (10) we can write their utility for deviating to $\hat{x}_A^2 = \bar{x}_k + \epsilon$ as:

$$\hat{U}_A^2(\cdot) = 1 + \beta\left[\frac{1}{2} - \frac{k}{4}(\bar{x}_k - \underline{x}_k)\right] = 1 + \beta\left[\frac{1}{2} - \frac{k^2}{4}\right], \tag{A18}$$

which in turn implies that deviating from $\mathbf{x^S}$ is profitable A^2 if:

$$1 + \beta\left[\frac{1}{2} - \frac{k^2}{4}\right] > \frac{\beta}{2} \Rightarrow \beta < \frac{4}{k^2}. \tag{A19}$$

.

Part 2: $d \leq k$ Note that as above candidate A^2's utility at $\mathbf{x^S}$ is simply $\frac{\beta}{2}$. In this case, their optimal deviation is to the platform $\hat{x}_A^2 = 2x_m^2 - \bar{x}_k + \epsilon$,[A4] which allows them to just barely win their regional contest while minimizing the organizational disunity they create. From (2) and (10) we can write their utility for deviating to $\hat{x}_A^2 = 2x_m^2 - \bar{x}_k + \epsilon$ as:

$$\hat{U}_A^2(\cdot) = 1 + \beta\left[\frac{1}{2} - \frac{k}{4}\Big((2x_m^2 - \bar{x}_k) - \underline{x}_k\Big)\right] = 1 + \beta\left[\frac{1}{2} - \frac{kd}{4}\right] \tag{A20}$$

which in turn implies that deviating is profitable if:

$$1 + \beta\left[\frac{1}{2} - \frac{kd}{4}\right] > \frac{\beta}{2} \Rightarrow \beta < \frac{4}{kd}. \tag{A21}$$

Note that at the value $k = d$ the conditions from (A20) and (A22) are identical, while when $d > k$ then $\max\{\frac{4}{k^2}, \frac{4}{kd}\} = \frac{4}{k^2}$ while if $d < k$ then $\max\{\frac{4}{k^2}, \frac{4}{kd}\} = \frac{4}{kd}$, which leads to the condition stated in Proposition 3. The analysis is identical for candidate B^1 **QED**

Corollary A2 *For the case $d \leq k$, if $\mathbf{x^S}$ is an equilibrium, then so is any strategy vector such that $\underline{x}_k < x_A^N < x_m^1$, $x_m^2 < x_B^N < \bar{x}_k$, and $1 - x_A^N = x_B^N - 1$.*

This Corollary tells us that if $\mathbf{x^S}$ is an equilibrium, then so is any Symmetric Polarization strategy vector at which: a.) x_A^N and x_B^N are more extreme than their regional medians, and b.) x_A^N and x_B^N are symmetric around the national median. It follows from the fact that the deviation payoff for A^2 at any such strategy vector is identical to that in (A20), with the exception that the $\bar{x}_k$ term is replaced x_B^N and the $\underline{x}_k$ term is replaced x_A^N. Since via symmetry $1 - x_A^N = x_B^N - 1$ which

[A4]The value $\hat{x}_A^2 = 2x_m^2 - \bar{x}_k + \epsilon$ comes from the fact that at $\mathbf{x^S}$ the distance between x_B^2 and x_m^2 is $(\bar{x}_k - x_m^2)$, and in order to be just barely closer to x_m^2 than B^2, A^2 would need to choose $\hat{x}_A^2 = x_m^2 - (\bar{x}_k - x_m^2) + \epsilon = 2x_m^2 - \bar{x}_k + \epsilon$.

means $x_A^N = 2 - x_B^N$, these terms fall out of (A20), and the final deviation condition captured in (A21) is identical. **QED**

B.1.1 Thresholds for Type 1 Nash Equilibrium

Propositions 2 and 3 identify the conditions under which $\mathbf{x^R}$ and/or $\mathbf{x^S}$ are equilibria. We now demonstrate that, from among all possible Type 1 strategy vectors, either $\mathbf{x^R}$ or $\mathbf{x^S}$ provides a threshold on the parameter β below which Type 1 Nash Equilibria do not exist. In the following proofs, we know that neither A^N nor B^N will have a profitable deviation from the analyzed strategy vectors, as we confine our study to the range $x_B^N, x_A^N \in [\underline{x}_k, \bar{x}_k]$. Furthermore, since we are studying only Type 1 strategy vectors, we know that neither A^1 nor B^2 will have a profitable deviation from any of the analyzed strategy vectors.

As such, the equilibrium relevant candidates in the following proofs are once again A^2 and B^1, who lose their regional contest in any Type 1 strategy vector. Without loss of generality, we analyze candidate A^2's deviation; the analyses are identical and symmetric for candidate B^1. Begin with the case in which $d > k$, meaning that $\underline{x}_k < x_m^1$ and $x_m^2 < \bar{x}_k$ such that $\mathbf{x^R}$ is not an equilibrium (Proposition 2).

Lemma A4 *Assume $d > k$. For any value of β, if $\mathbf{x^S}$ is not an equilibrium then there is no Type 1 Party Unity Nash Equilibrium.*

Proof of Lemma A4

For the case $d > k$, at any Party Unity strategy vector such that $x_B^N, x_A^N \in [\underline{x}_k, \bar{x}_k]$, A^2's optimal deviation is to the platform $\hat{x}_A^2 = x_B^2 + \epsilon$, which allows them to win their regional contest while minimizing the organizational disunity they create. Recalling that $\hat{x}_A^2 \approx x_B^2 = x_B^N$, this deviation creates additional disunity in party A of (approximately) $(x_B^N - x_A^N)$, and will be profitable for A^2 if:

$$1 + \hat{\pi}_A^N(\cdot)\beta > \pi_A^N(\mathbf{x})\beta \Rightarrow \beta < \frac{1}{\pi_A^N(\mathbf{x}) - \hat{\pi}_A^N(\cdot)} \Rightarrow \beta < \frac{4}{k(x_B^N - x_A^N)}. \quad \text{(A22)}$$

.

We know that $\frac{4}{k(x_B^N - x_A^N)}$ decreases as $(x_B^N - x_A^N)$ increases, which means that of all the possible Party Unity strategy vectors in the range $[\underline{x}_k, \bar{x}_k]$, the one at which A^2's deviation condition is the hardest to satisfy, in terms of the requirement on β, is that at which $x_A^N = \underline{x}_k$ and $x_B^N = \bar{x}_k$. In turn, for any β if A^2 deviates from $\mathbf{x^S}$ then she also deviates from any other strategy vector where $x_A^N, x_B^N \in [\underline{x}_k, \bar{x}_k]$, thus establishing Lemma A4. **QED**

For the case in which $d > k$ the value $\frac{4}{k^2}$ thus represents a threshold value. For any $\beta < \frac{4}{k^2}$ there will be no Type 1 Nash Equilibrium, and when $\beta = \frac{4}{k^2}$ only $\mathbf{x^S}$, will be the only Type 1 Nash Equilibria. As β increases beyond $\frac{4}{k^2}$ so does the range of Type 1 Nash Equilibria. The size of this range is determined by (A22): for $\beta > \frac{4}{k^2}$ a Type 1 strategy vector will be an equilibrium as long as x_B^N and x_A^N are such that (A22) is satisfied. As β gets larger, so does the range of equilibria; and when $\beta \to \infty$ any Type 1 strategy vector such that $x_A^N, x_B^N \in [\underline{x}_k, \bar{x}_k]$ can be sustained in equilibrium.

Now move to the case in which $d \leq k$, which means that $x_m^1 \geq \underline{x}_k$ and $x_m^2 \leq \bar{x}_k$ and that $\mathbf{x^R}$ will be an equilibrium as long as $\beta \geq \frac{2}{kd}$ (Proposition 2).

Lemma A5 *Assume $d \leq k$. For any value of β, if $\mathbf{x^R}$ is not an equilibrium then there is no Type 1 Party Unity Nash Equilibrium.*

Proof of Lemma A5

Part 1: Begin by comparing $\mathbf{x^R}$ to Party Unity strategy vectors at which $x_A^N > x_m^1$ and $x_B^N < x_m^2$. From above we know that the deviation condition for A^2 in this case is captured by (A22): $\beta < \frac{4}{k(x_B^N - x_A^N)}$. If $x_A^N > x_m^1$ and $x_B^N < x_m^2$ then $d > (x_B^N - x_A^N)$, meaning that $\frac{2}{kd} < \frac{4}{k(x_B^N - x_A^N)}$: the deviation condition from (A17) is harder to satisfy than that from (A22), i.e. $\mathbf{x^R}$ is sustainable as an equilibrium at a wider range of β than any Party Unity strategy vector such that $x_A^N > x_m^1$ and $x_B^N < x_m^2$.

Part 2: Now consider any Type 1 strategy vector such that $x_A^N \geq x_m^1$ and $x_m^2 < x_B^N \leq \bar{x}_k$. In this case, A^2's optimal deviation is to the platform $\hat{x}_A^2 = 2x_m^2 - x_B^2 + \epsilon$, which allows them to just barely win their regional contest while minimizing the organizational disunity they create (see ftn A4). Recalling that $\hat{x}_A^2 \approx 2x_m^2 - x_B^2$ and $x_B^2 = x_B^N$, this deviation creates additional disunity in party A of $(2x_m^2 - x_B^N - x_A^N)$, and using the same basic mechanics developed in the proofs of Proposition 3 we can show that this deviation will be profitable for A^2 if:

$$1 + \hat{\pi}_A^N(\cdot)\beta > \pi_A^N(\mathbf{x})\beta \Rightarrow \beta < \frac{1}{\pi_A^N(\mathbf{x}) - \hat{\pi}_A^N(\cdot)} \Rightarrow \beta < \frac{4}{k(2x_m^2 - x_B^N - x_A^N)}. \quad \text{(A23)}$$

In turn, it is straight-froward to show that $\frac{2}{kd} < \frac{4}{k(2x_m^2 - x_B^N - x_A^N}$: the deviation condition from (A17) is harder to satisfy than that from (A23), since the numerator of (A17) is smaller and the denominator of (A17) is larger.[A5] Thus, $\mathbf{x^R}$ is sustainable as an equilibrium at a wider range of β than at any Type 1 strategy vector such that $x_A^N \geq x_m^1$ and $x_m^2 < x_B^N \leq \bar{x}_k$.

Part 3: Now consider any Type 1 strategy vector such that $\underline{x}_k \leq x_A^N < x_m^1$ and $x_B^N \geq x_m^2$. An identical proof to that developed in Part 2, this time studying B^1's deviation condition, demonstrates that $\mathbf{x^R}$ is sustainable for a wider range of β than any such Type 1 strategy vector.

Part 4: To complete the proof of Lemma A5 consider a Type 1 strategy vector such that $\underline{x}_k \leq x_A^N < x_m^1$ and $x_m^2 < x_B^N \leq \bar{x}_k$, and without loss of generality assume that $(x_B^N - x_m^2) \geq (x_m^1 - x_A^N)$, such that A^N is (weakly) closer to the national median voter than B^N. In this case, A^2's optimal deviation is to the platform $\hat{x}_A^2 = 2x_m^2 - x_B^2 + \epsilon$ (see ftn A4), which yields the same deviation condition as (A23) above. In turn, it is straight-froward to show that $\frac{2}{kd} < \frac{4}{k(2x_m^2 - x_B^N - x_A^N)}$: the deviation condition from (A17) is harder to satisfy than that from (A23) since the numerator of (A17) is smaller and the denominator of (A17) is larger.[A6] Thus, $\mathbf{x^R}$

[A5]To see the latter, note that $k(2x_m^2 - x_B^N - x_A^N) > k^2$ can be simplified to $2x_m^2 - x_B^N - x_A^N > k$ and that this condition is easiest to satisfy when x_B^N and x_A^N adopt their smallest possible values, which in the case that we are studying will be $x_B^N = x_m^2 + \epsilon$ (since $x_B^N > x_m^2$) and $x_A^N = x_m^1$ (since $x_A^N \geq x_m^1$). Substituting into $2x_m^2 - x_B^N - x_A^N > k$ yields $d - \epsilon > k$, which in this case cannot be true since by construction $k \geq d$. Thus, the statement $k(2x_m^2 - x_B^N - x_A^N) > k^2$ leads to a contradiction and cannot be true, i.e. the denominator of (A17) is greater than the denominator of (A23).

[A6]To see the latter, note that $k(2x_m^2 - x_B^N - x_A^N) > k^2$ can be simplified to $2x_m^2 - x_B^N - x_A^N > k$ and that this condition is easiest to satisfy when x_A^N adopts its smallest possible value, which in the case that is sustainable as an equilibrium a wider range of β than any Type 1 strategy vector such that $\underline{x}_k \leq x_A^N < x_m^1$ and $x_m^2 < x_B^N \leq \bar{x}_k$. This establishes Lemma A5. **QED**

For the case in which $d \leq k$ the value $\frac{2}{kd}$ thus represents a threshold value: for any value $\beta < \frac{2}{kd}$ there will be no Type 1 Nash Equilibrium. Note that when $d \leq k$ then the relevant equilibrium condition for $\mathbf{x}^{\mathbf{S}}$ comes from (A21): candidate A^2 deviates if $\beta < \frac{4}{kd}$, and conversely $\mathbf{x}^{\mathbf{S}}$ is an equilibrium if $\beta \geq \frac{4}{kd}$. We now show that, when $d \leq k$, for any β in the range $\left[\frac{2}{kd}, \frac{4}{kd}\right)$, $\mathbf{x}^{\mathbf{R}}$ is the unique Type 1 Nash Equilibrium.

Lemma A6 *Assume $d \leq k$. For any value of β, if $\mathbf{x}^{\mathbf{S}}$ is not an equilibrium, then the only possible Type 1 Party Unity Nash Equilibrium is $\mathbf{x}^{\mathbf{R}}$.*

The proof of Lemma A6 proceeds identically to that of Lemma A5, establishing that $\mathbf{x}^{\mathbf{S}}$ can be sustained for a wider range of β than any other Type 1 strategy vector *except* $\mathbf{x}^{\mathbf{R}}$. It is omitted for reasons of redundancy, but available upon request.

For the case in which $d \leq k$ the value $\frac{4}{kd}$ from Proposition 3 thus represents a second threshold value. For $\frac{2}{kd} \leq \beta < \frac{4}{kd}$ the only Type 1 Nash Equilibrium is $\mathbf{x}^{\mathbf{R}}$. When $\beta = \frac{4}{kd}$ the strategy vector $\mathbf{x}^{\mathbf{S}}$ becomes an equilibrium, along with any other Symmetric Polarization strategy vector (Corollary A2). As β increases beyond $\frac{4}{kd}$ so does the range of Type 1 Nash Equilibria. The size of this range is determined by (A22) and (A23): for $\beta > \frac{4}{kd}$ Party Unity strategy vectors will be equilibria as long as $\hat{x}_A^2$ and x_A^N are such that the relevant condition (A22) or (A23) is satisfied. As β gets larger so does the range of equilibria, and when $\beta \to \infty$ any Type 1 strategy vector in the range $[\underline{x}_k, \bar{x}_k]$ can be sustained in equilibrium.

B.2 Type II Party Unity Nash Equilibria

Without loss of generality we consider cases in which candidate B^1 wins in region 1 and candidate B^2 wins in region 2. Given that we are studying only Party Unity strategy vectors, we know that by construction $x_B^1 = x_B^N \geq 1$, which means that a 'Type 2' strategy vector is only possible if $x_A^1 = x_A^N < x_m^1$ (otherwise A^1 by construction wins or ties in region 1). Thus, a 'Type 2' strategy vector requires $x_A^1 < x_m^1$ and $(x_m^1 - x_A^1) > (x_B^1 - x_m^1)$ (the condition for B^1 winning).

Lemma A7 *A 'Type 2' Party Unity strategy vector can only be a Nash Equilibrium if $d < \frac{k}{2}$.*

Proof: For B^1 to win in region 1 requires $(x_B^1 - x_m^1) < (x_m^1 - x_A^1)$. From all possible positions $x_B^1 = x_B^N \geq 1$, this inequality is easiest to be satisfy is when $x_B^1 = 1$, since this is the closest we can make x_B^1 to x_m^1. When $x_B^1 = 1$, in order for B^1 to defeat A^1 we need $(1 - x_m^1) < (x_m^1 - x_A^1) \Rightarrow x_A^1 < 2x_m^1 - 1$. In turn, if $\underline{x}_k \geq 2x_m^1 - 1$ then via Lemma 3 we know that candidate A^N deviates from any strategy vector at which B^1 defeats A^1 in region 1, since for B^1 to defeat A^1 we need $x_A^1 < 2x_m^1 - 1$. In words, in order for B^1 to defeat A^1, $x_A^1 = x_A^N$ has to be so extreme that it is outside boundary for non-deviation identified in Lemma 3. We

we are studying ($x_A^N < x_m^1$, $x_B^N > x_m^2$ and $(x_B^N - x_m^2) \geq (x_m^1 - x_A^N)$) will be $x_A^N = 2 - x_B^N$. Substituting $x_A^N = 2 - x_B^N$ into $2x_m^2 - x_B^N - x_A^N > k$ and rearranging yields $d > k$, which is never satisfied in this case since by construction $k > d$ and $x_A^N \geq x_m^1$. Thus, the statement $k(2x_m^2 - x_B^N - x_A^N) > k^2$ leads to a contradiction and cannot be true, i.e. the denominator of (A17) is larger than the denominator of (A23).

can rewrite $\underline{x}_k < 2x_m^1 - 1$ as $1 - x_m^1 < x_m^1 - \underline{x}_k$, and then substitute using the facts that $1 - x_m^1 = \frac{d}{2}$ and $x_m^1 - \underline{x}_k = \frac{k-d}{2}$, allowing to us to derive the final condition $d < \frac{k}{2}$. Via symmetry the same is true for the case in which candidates A^1 and A^2 win the respective regional elections. **QED**

Continuing with an example at which B^1 and B^2 win the regional contests, the following Lemma tells us that from among A's regional candidates, A^1 is the equilibrium relevant candidate.

Lemma A8 *If A^1 does not have a profitable deviation from a Type 2 strategy vector, then this strategy vector is a Nash Equilibrium.*

Proof: A^2's optimal platform deviation from a Type 2 strategy vector is to $\hat{x}_A^2 = x_B^2 + \epsilon$, at which they just barely win their regional contest at minimal cost in terms of organizational disunity. Recalling that at any Type 2 strategy vectors $x_B^2 = x_B^N$ and $x_A^2 = x_A^N$, the additional disunity created in party A by this deviation is (approximately) $(x_B^N - x_A^N)$, and the deviation is optimal if:

$$\beta < \frac{4}{k(x_B^N - x_A^N)}. \tag{A24}$$

Similarly, note that A^1's optimal deviation from a Type 2 strategy vector is to the platform $\hat{x}_A^2 = 2x_m^1 - x_B^1 + \epsilon$, and the additional disunity created in party A by this deviation is (approximately) $(2x_m^1 - x_B^N - x_A^N)$. The deviation is optimal if:

$$\beta < \frac{4}{k(2x_m^1 - x_B^N - x_A^N)}. \tag{A25}$$

In turn, it is straightforward to see that $\frac{4}{k(x_B^N - x_A^N)} < \frac{4}{k(2x_m^1 - x_B^N - x_A^N)}$, since by construction $x_B^N > 2x_m^1 - x_B^N$.[A7] In words, the deviation condition in for A^2 in (A24) is harder to meet than the deviation condition for A^1 in (A25), i.e. the optimal deviation is more costly for A^2 than it is for A^1. As a result, there will always exist a range of β for which this deviation is profitable for candidate A^1 but not for candidate A^2; and conversely there will never exist a range of β for which this deviation is profitable for A^2 but is not profitable for A^1. **QED**

Proof of Proposition 4

Following the logic from the proof of Lemma A7, candidate A^1's optimal deviation from $\mathbf{x}^{\underline{\mathbf{Y}}}$ is to the platform $\hat{x}_A^1 = 2x_m^1 - x_1 + \epsilon$. Recalling that $x_A^N = \underline{x}_k \equiv 1 - \frac{k}{2}$ at $\mathbf{x}^{\underline{\mathbf{Y}}}$, the additional disunity created in party A by this deviation is (approximately) $(2x_m^1 - 1 - (1 - \frac{k}{2}) = \frac{k}{2} - 2(1 - x_m^1)$. In turn, this deviation is profitable for A^1 if:

$$\hat{U}_A^1(\cdot) > U_A^1(\mathbf{x}^{\underline{\mathbf{Y}}}) \Rightarrow \beta > \frac{8}{k^2 - 4k(1 - x_m^1)} \equiv \frac{8}{k(k-2d)}. \tag{A26}$$

Therefore, if $d < \frac{k}{2}$ and $\beta \geq \frac{8}{k(k-2d)}$, then $\mathbf{x}^{\underline{\mathbf{Y}}}$ is an equilibrium. An identical analysis applies to candidate B^2's deviation from $\mathbf{x}^{\overline{\mathbf{Y}}}$, thus establishing the Proposition. **QED**

[A7] $x_B^N > 2x_m^1 - x_B^N \Rightarrow 2x_B^N > 2x_m^1 \Rightarrow x_B^N > x_m^1$ which is true by construction at any Type 2 strategy vector since $x_B^2 = x_B^N \geq 1 > x_m^1$.

Note that as long as $d < \frac{k}{2}$ the denominator of $\frac{8}{k(k-2d)}$ is greater than 0. There is an asymptote at the value $d = \frac{k}{2}$; for values $d > \frac{k}{2}$ Proposition 4's deviation conditions no longer apply, as candidate A^1 defeats candidate B^1 at $\mathbf{x}^{\underline{\mathbf{Y}}}$.

Proposition 4 identifies the conditions under which the Asymmetric Polarization strategy vectors $\mathbf{x}^{\underline{\mathbf{Y}}}$ and $\mathbf{x}^{\overline{\mathbf{Y}}}$ are equilibria. We now demonstrate that, from among all possible Type 2 strategy vectors, these Asymmetric Polarization strategy vectors provides a threshold level on the parameter β below which Type 2 Nash Equilibria do not exist. The proof follows those of Lemma A4-A6, demonstrating that of all Type 2 strategy vectors, $\mathbf{x}^{\underline{\mathbf{Y}}}$ and $\mathbf{x}^{\overline{\mathbf{Y}}}$ are the 'easiest' to sustain in equilibrium, as they maximize the party disunity that candidates A^1 and B^2, respectively, must create in order to just barely win their local elections.

Lemma A9 *If* $\mathbf{x}^{\underline{\mathbf{Y}}}$ *and* $\mathbf{x}^{\bar{\mathbf{Y}}}$ *are not Nash Equilibria then there is no Type 2 Party Unity Nash Equilibrium.*

Proof of Lemma A9

From above we know that A^1's optimal deviation from a Type 2 strategy vector is profitable if (A25) is satisfied. The lowest that the RHS of (A25) can be made, i.e. the easiest possible case for equilibrium, is when x_B^N and x_A^N adopt their lowest possible values, which are $x_B^N = 1$ and $x_A^N = \underline{x}_k$, which is precisely $\mathbf{x}^{\underline{\mathbf{Y}}}$. Thus, if β is small enough that A^1 chooses to deviate from $\mathbf{x}^{\underline{\mathbf{Y}}}$, then it is also small enough that A^1 would choose to deviate from any Type 2 strategy vector. **QED**

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Proof of Corollary 1

If $d > k$ we know five things: a) $\mathbf{x}^{\mathbf{R}}$ is not an equilibrium (Proposition 2), b) $\mathbf{x}^{\mathbf{S}}$ is an equilibrium if $\beta > \frac{4}{k^2}$ (Proposition 3), c) $\mathbf{x}^{\mathbf{S}}$ is the 'easiest' strategy vector to sustain as a Type 1 equilibrium (Lemma A4), d) no Type 2 strategy vector can be an equilibrium (Lemma A7), and e) $\frac{4}{kd} < \frac{4}{k^2}$. In turn, we know that if $\beta > \frac{4}{kd}$ then $\mathbf{x}^{\mathbf{G}}$ is not an equilibrium (Proposition 1), and that if $\beta < \frac{4}{k^2}$ then $\mathbf{x}^{\mathbf{S}}$ is not an equilibrium (Proposition 3). Since $\mathbf{x}^{\mathbf{S}}$ is the 'easiest' strategy vector to sustain as a Type 1 equilibrium, this implies that in this range no other Type 1 strategy vector is an equilibrium. Combine with (d), this establishes non-existence.

Proof of Corollary 2

Regional Medians $\mathbf{x}^{\mathbf{R}}$ is an equilibrium if $d > \frac{2}{k\beta}$. In turn, since $d \leq 1$ by construction, if $\frac{2}{k\beta} > 1$ then it is impossible for the condition $d > \frac{2}{k\beta}$ to hold, and $\mathbf{x}^{\mathbf{R}}$ cannot be an equilibrium. The inequality $\frac{2}{k\beta} > 1$ can be rewritten as $\beta > \frac{2}{k}$. Symmetric Polarization $\mathbf{x}^{\mathbf{S}}$ is an equilibrium if $d > \frac{4}{k\beta}$. In turn, since $d \leq 1$ by construction, if $\frac{4}{k\beta} > 1$ then it is impossible for the condition $d > \frac{4}{k\beta}$ to hold, and $\mathbf{x}^{\mathbf{S}}$ cannot be an equilibrium. The inequality $\frac{4}{k\beta} > 1$ can be rewritten as $\beta > \frac{4}{k}$. Asymmetric Polarization $\mathbf{x}^{\underline{\mathbf{Y}},\overline{\mathbf{Y}}}$ is an equilibrium if $d < \frac{k}{2} - \frac{4}{k\beta}$. In turn, since $d \geq 0$ by construction, if $\frac{k}{2} - \frac{4}{k\beta} < 0$ then it is impossible for the condition $d < \frac{k}{2} - \frac{4}{k\beta}$ to hold, and $\mathbf{x}^{\underline{\mathbf{Y}},\overline{\mathbf{Y}}}$ cannot be an equilibrium. The inequality $\frac{k}{2} - \frac{4}{k\beta} < 0$ can be rewritten as $\beta > \frac{8}{k^2}$. Since $k \leq 1$ we know that $\frac{2}{k} < \frac{4}{k} < \frac{8}{k^2}$, which means that if $\beta < \frac{2}{k}$, then neither $\mathbf{x}^{\mathbf{R}}$ nor $\mathbf{x}^{\mathbf{S}}$ nor $\mathbf{x}^{\underline{\mathbf{Y}},\overline{\mathbf{Y}}}$ can be an equilibrium. **QED**

Proof of Corollary 3

Beginning with the first bullet, we know from the proof of Corollary 2 above that if $\beta > \frac{8}{k^2}$ then $\mathbf{x}^{\mathbf{R}}$, $\mathbf{x}^{\mathbf{S}}$, and $\mathbf{x}^{\underline{\mathbf{Y}},\overline{\mathbf{Y}}}$ will all be equilibria for the appropriate values of d. Recall that the threshold above which $\mathbf{x}^{\underline{\mathbf{Y}},\overline{\mathbf{Y}}}$ will disappear as equilibria is $d = \frac{k}{2} - \frac{4}{k\beta}$, and the threshold above which $\mathbf{x}^{\mathbf{R}}$ will appear as an equilibrium is $d = \frac{2}{k\beta}$. The inequality $\frac{2}{k\beta} > \frac{k}{2} - \frac{4}{k\beta}$ can be rewritten as $\beta < \frac{12}{k^2}$. Thus as long as $\frac{8}{k^2} < \beta < \frac{12}{k^2}$, the value of d at which $\mathbf{x}^{\underline{\mathbf{Y}},\overline{\mathbf{Y}}}$ disappears as an equilibrium will come before the value of d at which $\mathbf{x}^{\mathbf{R}}$ becomes an equilibrium, establishing the non-monotonicity.

Moving to the second bullet, we know from the proof of Corollary 2 above that if $\beta > \frac{2}{k}$ then $\mathbf{x}^{\mathbf{R}}$ will be an equilibrium for the appropriate values of d; and that if $\beta < \frac{4}{k}$ then neither $\mathbf{x}^{\mathbf{S}}$ nor $\mathbf{x}^{\underline{\mathbf{Y}},\overline{\mathbf{Y}}}$ can be an equilibrium. Thus, via Proposition 2 we know that below the value $d = \frac{2}{k\beta}$ and above the value $d = k$ only $\mathbf{x}^{\mathbf{G}}$ will be an equilibrium; while for $d \in (\frac{2}{k\beta}, k)$ both $\mathbf{x}^{\mathbf{G}}$ and $\mathbf{x}^{\mathbf{R}}$ will be equilibrium, thus establishing the non-monotonicity. **QED**